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Edited by C. A. Brebbia and S. A. Orszag

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Th. V. Hromadka II, Ch.-Ch. Yen
G. F. Pinder

The Best Approximation Method
An Introduction

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The Best Approximation Method
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PREFACE

The most commonly used numerical techniques in solving engineering and mathematical models are the Finite Element, Finite Difference, and Boundary Element Methods. As computer capabilities continue to improve in speed, memory size and access speed, and lower costs, the use of more accurate but computationally expensive numerical techniques will become attractive to the practicing engineer.

This book presents an introduction to a new approximation method based on a generalized Fourier series expansion of a linear operator equation. Because many engineering problems such as the multi-dimensional Laplace and Poisson equations, the diffusion equation, and many integral equations are linear operator equations, this new approximation technique will be of interest to practicing engineers. Because a generalized Fourier series is used to develop the approximator, a "best approximation" is achieved in the "least-squares" sense; hence the name, the Best Approximation Method.

This book guides the reader through several mathematics topics which are pertinent to the development of the theory employed by the Best Approximation Method. Working spaces such as metric spaces and Banach spaces are explained in readable terms. Integration theory in the Lebesque sense is covered carefully. Because the generalized Fourier series utilizes Lebesque integration concepts, the integration theory is covered through the topic of converging sequences of functions with respect to measure, in the mean (Lp), almost uniformly

and almost everywhere. Generalized Fourier theory and linear operator theory are treated in Chapters 3 and 4. Chapters 5 and 6 present the Best Approximation Method, including several worked examples. Chapter 7 develops an application of the Best Approximation Method to complex analytic functions.

The book presents the several topics in a progressively advancing level in order to help carry the engineer from a calculus level mathematics background up through the more advanced mathematics level associated with the theory of Lebesgue integration and generalized Fourier series. Dozens of example problems are included which demonstrate or introduce additional pertinent concepts. Two FORTRAN computer programs are included which utilizes the Best Approximation Method towards solving potential problems using arbitrary basis functions, (Chpt. 6) and also using analytic basis functions (Chpt. 7) which provides an immediate extension of the Complex Variable Boundary Element Method.

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CHAPTER 1

WORKING SPACES

The following text introduces metric spaces, converging sequences, neighborhoods and other concepts which are necessary for the understanding of elementary approximation theory.

1.1. Metric Spaces

1.1.1. The Concept of a Metric

Let S be a set of objects (e.g., points x_j in the real number field **R**; functions x_j contained in the set of all continuous functions defined on the closed interval $[-\pi, \pi]$; integers x_j which are elements of the set of prime numbers; functions x_j which satisfy $\nabla^2 x_j = 0$ on the closed unit circle). Then S is said to be a metric space if a suitable metric or definition of the distance between each two elements of S is established. A distance function $d(x_i, x_j)$ qualifies as a metric on S if for each x_i, x_j in S the following axioms are satisfied:

(i) $d(x_i, x_j)$ is a real number.

(ii) $d(x_i, x_j) \geq 0$.

(iii) $d(x_i, x_j) = 0$ only when $x_i = x_j$.

(iv) $d(x_i, x_j) = d(x_j, x_i)$

(v) $d(x_i, x_j) \leq d(x_i, x_k) + d(x_k, x_j)$

1.1.2. Metrics

Typically, there are several candidates for the metric to use with a set of objects.

EXAMPLE 1.1.

Let $\hat{f}$ be an approximation of the function f over a domain $\mathcal{D}$ and suppose we are especially concerned in how accurately $\hat{f}$ approximates f at the two points x_1 and x_2 in $\mathcal{D}$. Let $e_1 = f(x_1) - \hat{f}(x_1)$ and $e_2 = f(x_2) - \hat{f}(x_2)$. Because we want $e(x) = f(x) - \hat{f}(x)$ to be as small as possible at the points x_1 and x_2, the measure of the approximation error is indicated by a value of distance between the ordered pairs (e_1, e_2) and $(0,0)$. A suitable metric is the Euclidean distance of $d_2((e_1, e_2), (0,0)) = (e_1^2 + e_2^2)^{\frac{1}{2}}$. Other choices for a metric include $d_\infty((e_1, e_2), (0,0)) = \max\{|e_1|, |e_2|\}$ or $d_1((e_1, e_2), (0,0)) = |e_1| + |e_2|$.

Notice that in this example, the set S is composed of two-dimensional vectors described by the ordered pair notation (x_1, x_2). Other choices for the approximation function $\hat{f}$ would typically result in different approximation errors (e_1, e_2) and the metric describes how well one approximation function succeeds in minimizing the approximation error at points x_1 and x_2 in $\mathcal{D}$.

Generally, the approximation error is evaluated at several points $x_1, x_2, \cdots, x_n$ in $\mathcal{D}$ resulting in n-dimensional vectors $(e_1, e_2, \cdots, e_n)$. The metric definitions considered in the above can be extended by $d_2(\underset{\sim}{e}, \underset{\sim}{\mathbf{0}}) = (e_1^2 + e_2^2 + \cdots + e_n^2)^{\frac{1}{2}}$; $d_\infty(\underset{\sim}{e}, \underset{\sim}{\mathbf{0}}) = \max\{|e_1|, |e_2|, \cdots, |e_n|\}$; $d_1(\underset{\sim}{e}, \underset{\sim}{\mathbf{0}}) = |e_1| + |e_2| + \cdots + |e_n|$ where vector notation is used to describe $\underset{\sim}{e} = (e_1, e_2, \cdots, e_n)$ and accordingly $\underset{\sim}{\mathbf{0}} = (0,0,\cdots,0)$. The subscript notation used in d_1, d_2 and d_∞ will be further discussed when l_p spaces are examined; however, it can be noted that above expressions for d_1, d_2 and d_∞

are the l_1, l_2 and l_∞ definitions for distance, respectively.

1.1.3. Metric Space Properties

Once a metric has been established over the set S, several concepts regarding open and closed neighborhoods of a point $x \in S$ follow analogous to the familiar concepts in Euclidean geometry. The following definitions are important in the development of the necessary approximation theory.

Definition 1.1: (Neighborhoods)

Let $\delta > 0$ and S be a metric space. Then a δ- neighborhood of the point $x \in S$ is the set of all points $y \in S$ such that $d(x,y)<\delta$. That is, $N_\delta(x) = \{y \in S: d(x,y) < \delta\}$. A closed δ-neighborhood of $x \in S$ is defined as $\bar{N}_\delta(x) = \{y \in S: d(x,y) \leq \delta\}$. A deleted neighborhood of $x \in S$ is defined as the set of all points $y \in S$ such that $0 < d(x,y) < \delta$.

The concepts of neighborhoods depends on the metric used to describe the distance between points. For example, in the usual two-dimensional Euclidean space with vectors described by the Cartesian plane coordinates (x,y), closed δ-neighborhoods of the origin having coordinates (0,0) is shown in Figure 1 for the l_1, l_2 and l_∞ definitions of distance, respectively.

1.1.4. Converging Sequences in a Metric Space

Definition 1.1 (Converging Sequence)

A sequence of vectors $x_1, x_2, \cdots$ in a metric space S is said to converge to a point x^* if the distance between x_n and x^* approaches zero as $n \to \infty$. That is, $x_n \to x^*$ if as $n \to \infty$, $d(x_n, x^*) \to 0$.

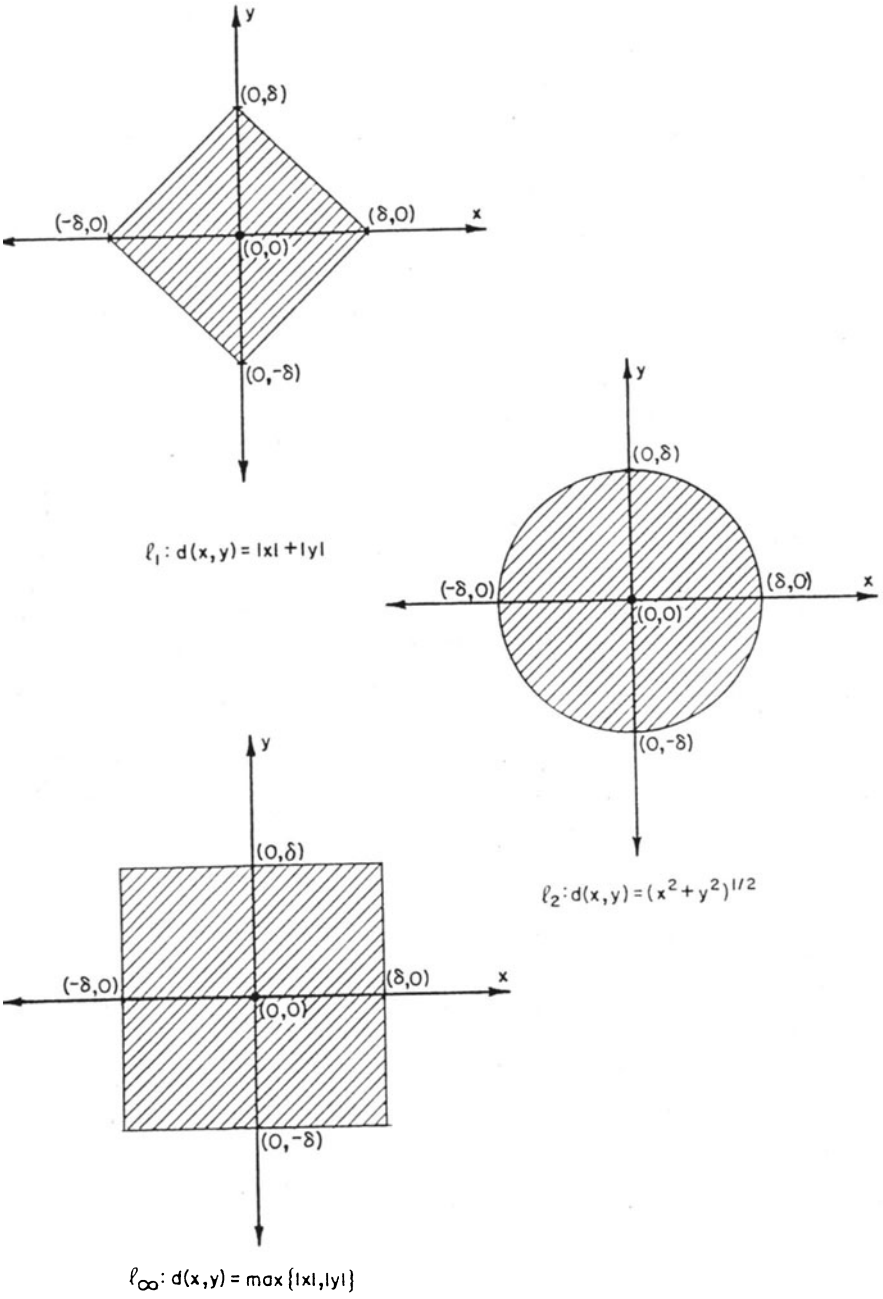


Fig. 1.1. Closed δ-Neighborhoods of Point (0,0) for ℓ_1, ℓ_2 and ℓ_∞ Definitions of Distance

EXAMPLE 1.2.

Let $x_n = n^2e^{-n}$ where n is a positive integer. Then the sequence $\{x_n\}$ tends to the limiting value of $x^* = 0$ as $n\to\infty$; that is $\lim_{n\to\infty} x_n = 0$. One method of proving that the above limit statement is true is to use an epsilon (ε) proof where it is shown that for any $\varepsilon > 0$ there exists a positive integer N (depending on ε) such that $d(x_n - x^*) < \varepsilon$ when $n \geq N$. For the subject sequence, let $d(x_n - x^*) = |x_n - x^*| = |n^2e^{-n}| = n^2e^{-n}$ for $n \geq 1$. If $\varepsilon = 0.001$, then a suitable N is N = 12. If various values of $n \leq 12$ are substituted into the formula for x_n it is seen that indeed $|x_n - x^*| < 0.001$. Should $\varepsilon = 10^{-10}$, a suitable N is N = 30. Because there exists an N for each ε, it is shown that $x_n \to x^*$ as $n\to\infty$. A more rigorous proof is to note that the Taylor series of $e^n = 1 + n + n^2/2! + n^3/3! + \cdots$ and therefore $n^2e^{-n} < n^2/(1 + n + n^2/2 + n^3/6) < 6/n$. Then for some $\varepsilon > 0$, setting $N > 6/\varepsilon$ guarantees that $n^2e^{-n} < \varepsilon$. Thus, for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$, proving that $n^2e^{-n} \to 0$ as $n\to\infty$.

EXAMPLE 1.3.

Suppose now that the sequence being studied $\{x_n\}$ is composed of two-dimensional vectors defined by $x_n = (ne^{-n}, n^2e^{-n})$. It is seen that $x_n \to x^* = (0,0)$ as $n\to\infty$. To prove this statement, it is noted that $ne^{-n} < 2/n$ and $n^2e^{-n} < 6/n$. Let the metric be the l_∞ distance $d_\infty(x_n, x_m) = \max\{|ne^{-n} - me^{-m}|, |n^2e^{-n} - m^2e^{-m}|\}$. Then $d_\infty(x_n, x^*) = \max\{ne^{-n}, n^2e^{-n}\} = n^2e^{-n}$. But from the previous one-dimensional example, $n^2e^{-n} < 6/\varepsilon$ implies $N(\varepsilon) \geq 6/\varepsilon$ for all $\varepsilon > 0$.

Thus for every $\varepsilon > 0$ there exists an N such that $n \geq N(\varepsilon) \rightarrow d_\infty(x_n, x^*) < \varepsilon$. Thus, $(ne^{-n}, n^2e^{-n}) \rightarrow (0,0)$ as $n \rightarrow \infty$.

Suppose that the l_1 distance of $d_1(x_n, x_m) = |ne^{-n} - me^{-m}| + |n^2e^{-n} - m^2e^{-m}|$ is used as the metric. Then to prove the limit statement we must show that as $n \rightarrow \infty$, $d_1(x_n, x^*) \rightarrow 0$. But $d_1(x_n, x^*) = ne^{-n} + n^2e^{-n} \leq 2n^2e^{-n} < 12/n$. Hence, for every $\varepsilon > 0$ there exists an $N(\varepsilon) = 12/\varepsilon$ such that $n \geq N(\varepsilon) \rightarrow d_1(x_n, x^*) < \varepsilon$. Again, $(ne^{-n}, n^2e^{-n}) \rightarrow (0,0)$ as $n \rightarrow \infty$.

Using the l_2 distance as the metric, $d_2(x_n, x_m) = ((ne^{-n} - me^{-m})^2 + (n^2e^{-n} - m^2e^{-m})^2)^{\frac{1}{2}} \leq (2(n^2e^{-n})^2)^{\frac{1}{2}} < \sqrt{2}\, n^2e^{-n} < 6\sqrt{2}/n$. Then for every $\varepsilon > 0$ there exists an $N(\varepsilon) \geq 6\sqrt{2}/\varepsilon$ such that $n \geq N(\varepsilon) \rightarrow d_2(x_n, x^*) < \varepsilon$.

EXAMPLE 1.4.

As a final illustration, consider a sequence of dimension 100 vectors defined by $x_n = (ne^{-n}, n^2e^{-n}, \cdots, n^{100}e^{-n})$. Then it can be shown that $x_n \rightarrow x^* = (0,0,\cdots,0)$ as $n \rightarrow \infty$ by noting that $n^{100}e^{-n} < 101!/n$. Thus for the l_1 distance as a metric, $d_1(x_n, x^*) = \sum_{j=1}^{100} n^j e^{-n} < 100n^{100}e^{-n} < (100)(101!)/n$.

Letting $N(\varepsilon) \geq (100)(101!)/\varepsilon$ shows the existence of $N(\varepsilon)$ for every $\varepsilon > 0$ in the limit criteria. For the l_∞ metric, $N(\varepsilon) \geq 101!/\varepsilon$. For the l_2 metric, $N(\varepsilon) = (10)(101!)/\varepsilon$.

The above examples show that the limit of a sequence is independent of the choice of the metric, and that multidimensional vectors follow the same limit criteria that is understood for the basic one-dimensional case.

We are interested in the above two concepts because the Best Approximation Method will be using multidimensional vectors to represent the approximation function components, and the best approximation will be developed using the l_2 definition for distance as the metric. That is, the Best Approximation Method will determine the best approximation function (from a set of basis functions) in the l_2 sense in that the l_2 distance between satisfying the given linear operator relationship and the boundary and initial conditions will be a minimum.

1.2. Linear Spaces

The subject of linear spaces and linear operators is fundamental to the development of the theory utilized in the Best Approximation Method. Therefore, the following five definitions must be thoroughly understood by the reader.

Definition 1.3: (Linear Space or Vector Space)

Let $S = \{x,y,z,\cdots\}$ be a set and $F = \{\alpha,\beta,\gamma,\cdots\}$ be a scalar field (e.g., the real number field, **R**). Also let there be defined the operations of addition between every two elements of S and scalar multiplication between every element of S and every element of F such that for all $x,y,z \in S$ and $\alpha, \beta \in F$ satisfy:

(i) $x + y \in S$.

(ii) $\alpha x \in S$.

(iii) $x + y = y + x$.

(iv) $(x + y) + z = x + (y + z)$.

(v) There exists a zero element in S, designated by **0**, such that for $x \in S$, $x + \mathbf{0} = x$.

(vi) For each $x \in S$, there also exists in S the negative of x (noted as -x) such that $x + (-x) = \mathbf{0}$.

(vii) $\alpha(\beta x) = (\alpha\beta)\, x$

(viii) $\alpha(x + y) = \alpha x + \alpha y$

(ix) $(\alpha + \beta)\, x = \alpha x + \beta x$

(x) $1x = x$

Then S is said to be a linear space over the field F. If F is the real number field, **R**, then S is a real linear space. The real linear space, S, is the focus for the development of the Best Approximation Method.

Definition 1.4: (Linear Combination)

Let $x_1, x_2, \cdots, x_n$ be elements of S and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be elements of **R**. The sum $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ is called a linear combination of the x_j.

Definition 1.5: (Linear Independence)

Let $x_1, x_2, \cdots x_n$ and $\alpha_1, \alpha_2 \cdots$ be elements of S and **R**, respectively. Then the x_j are called linearly independent if and only if the sum $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$ implies each α_j is zero. If there exists a set $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbf{R}$ such that not all of these elements are zero and yet the subject sum is zero, then the set of x_j are said to be linear dependent.

Definition 1.6: (Dimension of the Linear Space, S)

Let S be a linear space. Suppose that there exist n elements $x_1, x_2, \cdots, x_n$ in S which are linearly independent, and every set of n + 1 elements in S is linearly dependent. Then the dimension of S is said to be n, that is Dim (S) = n. If there exist m linearly independent elements in S for every integer m > 0, then Dim (S) = ∞.

Definition 1.7: (Basis for a Linear Space, S)

Let $x_1, x_2, \cdots, x_n$ be a linearly independent set of elements in S such that each x in S can be written as a linear combination of the x_j, j = 1,2,$\cdots$,n. Then the x_j are said to form a basis for S.

EXAMPLE 1.5.

An example of a linear space is the set $\mathbf{R}^m$ of all m-dimension vectors $(x_1, x_2, \cdots x_m)$ where each $x_j \in \mathbf{R}$. Addition, subtraction, and scalar multiplication are all defined in the usual way. The zero vector is defined by $\mathbf{0} = (0,0,\cdots,0)$. It follows immediately that Dim $(\mathbf{R}^m) = m$ and a basis for $\mathbf{R}^m$ is the m vectors $\{x_k\}$ where x_k is the zero vector with a 1 placed at the k-coordinate.

EXAMPLE 1.6.

An important example of a linear space is the set of all functions which are continuous over the closed interval [a,b]. This space is noted by C[a,b] and satisfies all the properties of Definition 1.3. For two vectors f_1 and f_2 in C[a,b], addition is defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ for each point x in [a,b], and scalar multiplication is defined for $\lambda \in \mathbf{R}$ by $\lambda f_1 = \lambda f_1(x)$.

The zero vector is defined by the continuous function $f_1(x) = 0$. Subtraction is defined by $(f_1 - f_2)(x) = f_1(x) - f_2(x)$. It is noted that the addition and subtraction of f_1 and f_2 is another continuous function, and the scalar multiplication of f_1 is a continuous function; that is, $f_1 + f_2 \in C[a,b]$, $f_1 - f_2 \in C[a,b]$, and $\lambda f_1 \in C[a,b]$.

A linear combination of vectors $f_j \in C[a,b]$ is a continuous function $f = \sum_{j=1}^{m} \lambda_j f_j \in C[a,b]$. However, the Dim $(C[a,b]) = \infty$ and the subject of converging sequences of functions must be introduced before an expansion of f in terms of basis functions can be shown to be equivalent.

1.3 Normed Linear Spaces

A generalization of the concept of distance between two vectors of a linear space is the norm.

Definition 1.8. (Norm)

A norm is a function $||x||$ defined for each $x \in S$ with the following properties:

(i) $||x|| \geq 0$

(ii) $||\lambda x|| = \lambda ||x||$ for $\lambda \in \mathbf{R}$

(iii) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in S$

(iv) $||x|| = 0$ implies $x = 0$

A linear space with an associated norm is called a normed linear space. A normed linear space is also a metric space and, therefore, the properties and concepts embodied in metric spaces (e.g., neighborhoods) also apply in a normed linear space.

EXAMPLE 1.7.

In the vector space $\mathbf{R}^m$, a suitable norm is the l_2 norm where for $x,y\in\mathbf{R}^m$ $||x-y|| = ((x_1-y_1)^2 + (x_2-y_2)^2+\cdots+(x_m-y_m)^2)^{\frac{1}{2}}$.

It is seen that the properties of Definition 1.8 are satisfied:

(i) $||x|| \geq 0$

(ii) $||\lambda x|| = ((\lambda x_1)^2+\cdots+(\lambda x_m)^2)^{\frac{1}{2}}$

$= (\lambda^2(x_1^2 + x_2^2 +\cdots+ x_m^2))^{\frac{1}{2}}$

$= \lambda\ (x_1^2 + x_2^2+\cdots+ x_m^2)^{\frac{1}{2}} = \lambda||x||$

(iii) $||x+y|| \leq ||x|| + ||y||$. In Euclidian one, two, and three-dimensional space where the norm is defined to be the usual Euclidian distance, the triangle inequality follows. For higher dimensions, or for other norms, Minkowski's inequality must be used as a proof.

(iv) $||x|| = 0$ implies $x_1^2 = x_2^2 =\cdots=x_m^2 = 0$, and $x = \mathbf{0}$.

EXAMPLE 1.8.

In C[a,b], a norm $||f||$ where $f\in C[a,b]$ can be defined as $||f|| = \max\{|f(x)|: x\in[a,b]\}$. Consequently for f_1 and f_2 in C[a,b], $||f_1-f_2|| = \max\{|f_1(x)-f_2(x)|: x\in[a,b]\}$. Figure I.2 shows the definition of $||f_1 - f_2||$.

1.4. Banach Spaces

Let S be a normed linear space and let $\{x_n\}$ be a sequence of vectors such that $x_n\in S$ for each n. Suppose that $x_n\to x^*$ as $n\to\infty$. Then if $x^*\in S$, the sequence $\{x_n\}$ converges to a vector x^* in S.

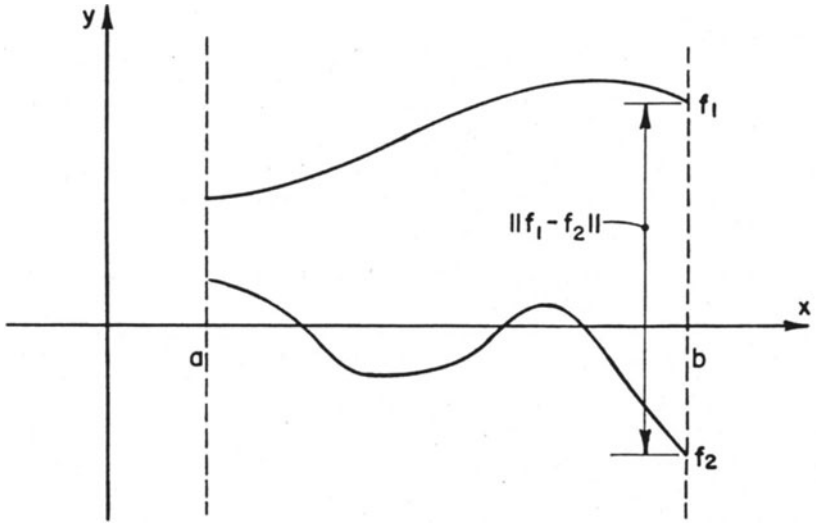


Fig. 1.2. Definition of $||f_1 - f_2||$ for $f_1, f_2 \in C[a,b]$

However, oftentimes a sequence can converge to a vector x^* but x^* is not an element of S. This can be illustrated by the following examples:

EXAMPLE 1.9.

Let S be the open interval (0,1) with the norm $||x|| = |x|$ for $x \in S$. Let $\{x_n\}$ be the sequence of vectors (points) defined by $x_n = \frac{1}{n}$ for each $n \geq 1$. Then $x_n \to x^* = 0$ as $n \to \infty$. But $0 \notin S$, hence $x_n \to x^* \notin S$ as $n \to \infty$.

EXAMPLE 1.10.

Let S be the set of all rational numbers, Q (i.e., the set of all real numbers q such that $q = m/n$ where m and n are integers). Let T be the set of all irrational numbers (i.e., $T = \mathbf{R}-Q$). Let t be an element of T and let x_n be the first n digits of the decimal expansion of t. Then each $x_n \in S$, and as $n \to \infty$ $x_n \to t \notin S$. Now consider $\{x_n\} = \{1, 1+1, 1+1+\frac{1}{2!}, 1+1+\frac{1}{2!}+\frac{1}{3!}, \cdots\}$ where $x_n = \sum_{j=0}^{n} 1/j!$. Then $x_n \to x^* = e$ as $n \to \infty$ (the term x_n is the first n terms of the Taylor series for e^x at $x = 1$). Each term x_n is in Q(i.e., S) and therefore $\{x_n\}$ is in Q. But $x_n \to x^* = e$ as $n \to \infty$ where $e \in T$ and $e \notin Q$. Thus, $x_n \to x^* \notin S$ as $n \to \infty$.

EXAMPLE 1.11.

As in EXAMPLE 1.10, let $S = Q$ and $T = \mathbf{R} - Q$. Then convergent sequences $\{x_n\}$ in S can be constructed which converge to vectors $x^* \in T$ for each $t \in T$; that is, the space Q is incomplete. In contrast, had $S = \mathbf{R}$, then each convergent sequence $\{x_n\}$ in $\mathbf{R}$ converges to a point $x^* \in \mathbf{R}$. The space $\mathbf{R}$ is said to be complete.

1.4.1. Cauchy Sequences

The definition of a convergent sequence given in Definition 1.2 is too general for practical purposes: the limit of the sequence, x^*, must be known to prove $\{x_n\} \to x^*$ as $n \to \infty$.

Another approach to evaluating whether a sequence converges is to show that for every $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $||x_n - x_{n+p}|| < \varepsilon$ for $n \geq N(\varepsilon)$ and $p = 1,2,\cdots$. That is, for any $n \geq N(\varepsilon)$, all of the remaining terms x_j of the sequence $(j > n)$ lie in the closed ε-neighborhood $\bar{N}_\varepsilon(x_n)$. As $\varepsilon \to 0$ the associated $N(\varepsilon)$ get larger, resulting in a parallel sequence of closed ε-neighborhoods where each successive $\bar{N}_{\varepsilon_{k+1}}(x_n)$ is contained in the previous $\bar{N}_\varepsilon(x_n)$ for $\varepsilon_k > \varepsilon_{k+1}$. This sequence of nested closed neighborhoods focuses about the limit vector x^* as $N(\varepsilon) \to \infty$.

A sequence $\{x_n\}$ which satisfies the above criteria for convergence is called a Cauchy sequence.

It is noted that whether the limit point x^* is an element of S is not established by simply showing the sequence $\{x_n\}$ is Cauchy.

1.4.2. Complete Normed Linear Space (Banach Space)

A normed linear space S where any Cauchy sequence $\{x_n\}$ has its limit x^* contained in S is said to be complete. A complete normed linear space is called a Banach space.

EXAMPLE 1.12.

The normed linear space $\mathbf{R}^m$ is a Banach space. This can be shown by noting that a sequence $\{x_n\}$ in $\mathbf{R}^m$ is composed of vectors of the form $(x_1, x_2, \cdots, x_n)$ where each coordinate x_j is in $\mathbf{R}$. A Cauchy sequence in $\mathbf{R}^m$ implies that each coordinate x_j of the vectors is also a Cauchy sequence which converges to its own limit point, x_j^* (the space $\mathbf{R}$ is complete). Thus each $\{x_j\} \to x_j^*$ as $n \to \infty$. Thus the vectors $\{x_n\} \to \{x_1^*, x_2^*, \cdots, x_m^*\} \in \mathbf{R}^m$ as $n \to \infty$. Hence, each Cauchy sequence in $\mathbf{R}^m$ converges to a limit which is an element of $\mathbf{R}^m$. Thus $\mathbf{R}^m$ is a Banach space.

EXAMPLE 1.13.

C[a,b] with the maximum norm is a Banach space. Let $\{f_n\}$ be a Cauchy sequence in C[a,b]. Using the norm defined in EXAMPLE 1.8 it is seen that for $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that $||f_n - f_{n+p}|| < \varepsilon$ for $n \geq N(\varepsilon)$ and $p = 1,2,\cdots$. Thus, $|f_n(x) - f_{n+p}(x)| < \varepsilon$ for $x \in [a,b]$. Let $x_o \in [a,b]$. Then $|f_n(x_o) - f_{n+p}(x_o)| < \varepsilon$ for $n \geq N(\varepsilon)$ and $p = 1,2,\cdots$. That is $\{f_n(x_o)\}$ is a Cauchy sequence in $\mathbf{R}$ and $f_n(x_o) \to f_n^*(x_o) \in \mathbf{R}$ as $n \to \infty$. Thus a limit $f_n^*(x)$ exists for $\{f_n(x)\}$ for all $x \in [a,b]$. Now we must see if $f_n^*(x)$ is continuous (i.e., show $f_n^*(x) \in C[a,b]$). To show $f_n^*(x)$ is continuous, we must show that a small δ-change in x (in [a,b]) results in a small change in $f_n^*(x)$. But $|f_n^*(x+\delta) - f_n^*(x)| = |f_n^*(x+\delta) - f_n(x+\delta) + f_n(x+\delta) - f_n(x) + f_n(x) - f_n^*(x)| \leq |f_n^*(x+\delta) - f_n(x+\delta)| + |f_n(x+\delta) - f_n(x)| + |f_n(x) - f_n^*(x)|$. The middle term $|f_n(x+\delta) - f_n(x)|$ reflects the

continuity of the terms f_n in the sequence and can be made arbitrarily small (say less than $\varepsilon^1/3$) by choosing δ small enough. The other two terms reflect the convergence of the two Cauchy sequences $\{f_n^*(x+\delta)\}$ and $\{f_n^*(x)\}$. Choosing $\varepsilon = \varepsilon^1/3$ determines $N(\varepsilon^1/3)$ such that $|f_n^*(x+\delta) - f_n(x+\delta)| + |f_n(x) - f_n^*(x)| < \varepsilon^1/3 + \varepsilon^1/3 = 2\varepsilon^1/3$. Thus, for any $\varepsilon^1 > 0$ there exists a $\delta > 0$ (and an $N(\varepsilon^1)$) such that $|f_n^*(x+\delta) - f_n^*(x)| < 3\varepsilon^1 = \varepsilon$. Since f_n^* is continuous, $f_n^* \in C[a,b]$. Thus a Cauchy sequence in C[a,b] contains a limit in C[a,b]. Therefore C[a,b] is a Banach space.

EXAMPLE 1.14

The space C[a,b] with the norm $||f|| = (\int_a^b f^2(x)dx)^{\frac{1}{2}}$ is not complete. This can be shown by example. Let [a,b] = [-1,1] and the sequence $\{f_n\}$ be in C[-1,1] where

$$f_n(x) = \begin{cases} -1 , & -1 \le x \le -1/n \\ nx , & -1/n < x < 1/n \\ 1 , & \frac{1}{n} \le x \le 1 \end{cases}$$

Using the defined norm $||f||$, $||f_n - f_{n+p}|| <$

$$(2\int_0^{1/n} (1-nx)^2dx)^{\frac{1}{2}} = (2\int_0^{1/n} (1-2nx+n^2x^2)dx)^{\frac{1}{2}}$$

$$= (1/6n)^{\frac{1}{2}}$$

Thus letting $N(\varepsilon) \ge 1/6\varepsilon^2$ gives that for every $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that $||f_n - f_{n+p}|| < \varepsilon$ for $n \ge N(\varepsilon)$ and $p = 1,2,\cdots$.

Thus, $\{f_n\}$ is a Cauchy sequence using the defined norm. However as $n\to\infty$, $f_n\to f^*$ where

$$f^*(x) = \begin{cases} -1 , & -1 \le x < 0 \\ 0 , & x = 0 \\ 1 , & 0 < x \le 1 \end{cases}$$

Thus $f_n\to f^*$ as $n\to\infty$, but f^* is not continuous over $[-1,1]$.

CHAPTER 2

INTEGRATION THEORY

2.0 Introduction

Before introducing the theory of generalized Fourier series, the key elements of integration theory need to be reviewed. This is important in order to understand the value of the approximation developed by the Best Approximation Method.

2.1 The Riemann and Lebesgue Integrals: Step and Simple Functions

The engineer is typically aware of the Riemann integration theory. To proceed, the characteristic function of the set E is the function χ_E defined by

$$\chi_E = \begin{cases} 1, \text{ if } x \text{ is in } E \\ 0, \text{ if } x \text{ is not in } E \end{cases} \tag{2.1}$$

A step function ϕ is a finite linear combination of characteristic functions of intervals,

$$\phi = \sum_{j=1}^{n} \lambda_j \chi_{E_j} \tag{2.2}$$

where $\lambda_j \in \mathbf{R}$. Since E_j are intervals, let E_j have endpoints a_j, b_j such that $b_j > a_j$. Then the Riemann integral of ϕ is given by

$$\int \phi = \sum_{j=1}^{n} \lambda_j (b_j - a_j) \tag{2.3}$$

The Lebesgue integral can be obtained analogously except that the characteristic functions are applied to sets E_j which are measurable with measure mE_j. Then for a simple function

$$\phi = \sum_{j=1}^{n} \lambda_j \chi_{E_j} \tag{2.4}$$

The Lebesgue integral of ϕ is given by

$$\int \phi = \sum_{j=1}^{n} \lambda_j \, m \, \chi_{E_j} \tag{2.5}$$

Although the Lebesgue and Riemann integrals often show many similarities, the differences in theory and application are significant enough that much of theory of generalized Fourier series become feasible upon the development of the Lebesgue integral.

2.2. Lebesgue Measure

From (2.4) and (2.5) it is seen that the Lebesgue integral utilizes sets E_j which are not necessarily intervals and which require that the measure mE_j exist in some sense.

Definition 2.1: (Lebesgue Outer Measure)

Let E be a set. The Lebesgue outer measure of E, denoted by $m_e(E)$ is the greatest lower bound of the lengths of all open sets θ such that $E \subset \theta$.

Definition 2.2: (Interior Measure)

The interior measure $m_i(E)$ of a set E is the least upper bound of the lengths of all closed sets C such that $C \subset E$.

If $m_e(E) = m_i(E)$, then the set E is said to be measurable. It can be shown that a set E is measurable if and only if for any $\varepsilon > 0$ there is an open set θ such that $E \subset \theta$ and $m_e(\theta - E) < \varepsilon$; similarly, there is a closed set $C \subset E$ such that $m_e(E - C) < \varepsilon$; finally, there are sets θ and C such that $C \subset E \subset \theta$ and $m_e(\theta - C) < \varepsilon$.

Several important theorems follow from the definition of measure:

(i) If a set E is measurable, then $\bar{E}$ is measurable.

(ii) If E_1 and E_2 are measurable sets, then the union $E_1 \cup E_2$ and the intersection $E_1 \cap E_2$ are measurable.

(iii) If E_1 and E_2 are measurable sets, then $m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$.

(iv) Let $E_1, E_2, E_3, \cdots$ be a sequence of measurable sets. Then $\bigcup_{n=1}^{\infty} E_n$ is measurable. If the E_n are mutually disjoint, $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$.

(v) Any open set θ in $\mathbf{R}$ can be written in terms of a countable union of disjoint open intervals θ_i called component intervals. Because each open interval $\theta_i = (a_i, b_i)$ has length $(b_i - a_i)$, the measure of any open set in $\mathbf{R}$ is the length of θ; that is, $m(\theta) = \sum_{i=1}^{\infty} m(\theta_i)$.

Definition 2.3: (Measure Zero)

If a set E has $m_e(E) = 0$, then $m(E) = 0$; that is, E has measure zero.

Definition 2.4: (Almost Everywhere, ae)

A property or function which applies everywhere on a set E except for a subset $E^1 \subset E$ such that $m(E^1) = 0$ is said to apply almost everywhere or ae.

2.3. Measurable Functions

Not only must the set E (over which the integration is to occur) have special characteristics (be measurable), the functions which are to be integrated over E must be qualified as well.

Definition 2.5: (Measurable Function)

Let E be a measurable set and the real function $\phi(x)$ be defined on E. Then ϕ is said to be a measurable function (or measurable) on E if for any $\alpha \in \mathbf{R}$ the set $\{x \in E: \phi(x) > \alpha\}$ is measurable.

Several modifications of Definition 2.6 are possible. The function ϕ is measurable on E if for any $\alpha \in \mathbf{R}$ the sets $\{x \in E: \phi(x) \geq \alpha\}$, $\{x \in E: \phi(x) \leq \alpha\}$, $\{x \in E: \phi(x) < \alpha\}$ are measurable. Or for $\alpha < \beta$ and $\alpha, \beta \in R$, ϕ is measurable if the sets $\{x \in E: \alpha < \phi(x) < \beta\}$, $\{x \in E: \alpha \leq \phi(x) \leq \beta\}$, and so forth, are measurable.

Several theorems indicate whether a function ϕ is measurable on the measurable set E:

(i) If $E^1 \subset E$, then ϕ is measurable on E^1.

(ii) A constant function ϕ is measurable for $\lambda \in \mathbf{R}$.

(iii) $(\lambda\phi)$, $(\phi + \lambda)$, and ϕ^2 are measurable on E.

(iv) Let ϕ_1 and ϕ_2 be measurable on E, then $(\phi_1 + \phi_2)$, $(\phi_1 - \phi_2)$, $(\phi_1\phi_2)$ and (ϕ_1/ϕ_2) are measurable on E where for ϕ_1/ϕ_2 it is assumed $\phi_2 \neq 0$.

(v) If ϕ_1 is measurable on E and $\phi_2 = \phi_1$ ae, then ϕ_2 is measurable on E.

(vi) A continuous function is measurable.

(vii) A function which is continuous over E except for a countable number of discontinuities, is measurable.

Fortunately, the class of functions normally dealt with in engineering studies and the sets involved are both measurable. However in studying whether the approximation function to the operator equation converges to the exact solution, the reliance upon integration theory becomes important as utilized in proving convergence of a generalized Fourier series.

2.4. The Lebesgue Integral

The Lebesgue integral is similar in concept to the Riemann integral in that a partition is utilized in the integration. In the Riemann integral, the domain is partitioned; in the Lebesgue integral, the range of the integrable function $\phi(x)$ is partitioned.

(ii) If $mE = 0$, then $\int_E \phi = 0$.

(iii) If $E = E_1 \cup E_2$ where all the sets are measurable and $E_1 \cap E_2 = \emptyset$, then $\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi$

(iv) Let E be a measurable set and ϕ be measurable on E.

Then

- $\int_E \lambda = \lambda m(E)$ for λ is a constant in $\mathbf{R}$
- $\int_E \lambda\phi = \lambda \int_E \phi$
- If $\alpha \leq \phi \leq \beta$, then $\alpha m(E) \leq \int \phi \leq \beta m(E)$

(v) Let ϕ_1 and ϕ_2 be bounded and measurable on a measurable set E. Then

- $\left| \int_E \phi_1 \right| \leq \int_E |\phi_1|$
- If $\phi_1 \leq \phi_2$, then $\int_E \phi_1 \leq \int_E \phi_2$
- $\int_E (\phi_1 + \phi_2) = \int_E \phi_1 + \int_E \phi_2$
- If $\phi_1 = \phi_2$ ae, then $\int_E \phi_1 = \int_E \phi_2$
- $\int_E \phi_1 = 0$ and $\phi_1 \geq 0$ ae on E, then $\phi_1 = 0$ ae on E

If $\phi(x)$ is a function which is Riemann integrable on $E = [a,b]$, then it is also Lebesgue integrable and the two integrals equal each other. However, a Lebesgue integrable function is not necessarily Riemann integrable.

2.4.1. Bounded Functions

Let $\alpha < \phi(x) < \beta$ be defined on $[a,b]$, such that α and β are in $\mathbf{R}$. Partition (α,β) into n intervals by points ξ_j such that $\alpha = \xi_o < \xi_1 < \cdots < \xi_{n-1} < \xi_n = \beta$. Define the sets E_j in the domain by $E_j = \{x \in [a,b]: \xi_{j-1} \leq \phi(x) < \xi_j\}$ for $j = 1,2,\cdots,n-1$, and $E_n = \{x \in [a,b]: \xi_{n-1} \leq \phi(x) \leq \xi_n\}$. Then the upper sum S and the lower sum s are given by

$$S = \sum_{j=1}^{n} \xi_j \, m(E_j)$$

$$s = \sum_{j=1}^{n} \xi_{j-1} \, m(E_j)$$

Let $\overline{\int_a^b} \phi(x)dx$ be the greatest lower bound of all possible values of S for all possible partitions, and let $\underline{\int_a^b} \phi(x)dx$ be the least upper bound of all possible values of s for all possible partitions. If the above two integrals are equal, then $\phi(x)$ is said to be Lebesgue integrable on $[a,b]$ and the integral is noted by $\int_a^b \phi(x)dx$ or simply $\int_a^b \phi$.

Several important theorems follow:

(i) Let E be a measurable set in $[a,b]$. Let ϕ be bounded and measurable on E. Then the Lebesgue integral exists and is noted by $\int_E \phi$. Also, $\int_E |\phi| \leq \infty$.

(vi) Let ϕ be bounded and measurable on $E = \bigcup_{j=1}^{\infty} E_j$ where the E_j are all measurable sets and are mutually disjoint. Then $\int_E \phi = \sum_{j=1}^{\infty} \int_{E_j} \phi$

(vii) Let ϕ_1 and ϕ_2 be bounded and measurable on $E = (a,b)$ and $\int_E (\phi_1 - \phi_2)^2 = 0$. Then $\phi_1 = \phi_2$ ae in E.

(viii) Let E be a measurable set and let $\langle \phi_j \rangle$, $j = 1,2,\cdots,n$ be bounded and measurable on E and λ_j, $j = 1,2,\cdots,n$ be constants in $\mathbb{R}$. Then $\int_E \sum_{j=1}^{n} \lambda_j \phi_j = \sum_{j=1}^{n} \lambda_j \int_E \phi_j$

2.4.2. Unbounded Functions

Section 2.4.1 addressed bounded functions and Lebesgue integration. In this section, unbounded functions are considered. Let E be a measurable set and let ϕ be divided into positive and negative parts by

$$\phi^+(x) = \begin{cases} \phi(x), & x \in E \text{ such that } \phi(x) \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi^-(x) = \begin{cases} \phi(x), & x \in E \text{ such that } \phi(x) < 0 \\ 0, & \text{otherwise} \end{cases}$$

Then ϕ^+ and ϕ^- are both nonnegative functions such that $\phi = \phi^+ - \phi^-$. Consider first ϕ^+. Define functions f_p by

$$f_p(x) = \begin{cases} \phi(x), & x \in E \text{ such that } \phi(x) \leq p \\ p, & \text{otherwise} \end{cases}$$

Then f_p is bounded, measurable, and Lebesgue integrable. The Lebesgue integral $\int_E \phi^+$ is defined by

$$\int_E \phi^+ = \lim_{p \to \infty} \int_E f_p$$

If the limit is finite, $\int_E \phi^+$ is said to exist or ϕ^+ is integrable on E. If $\int_E \phi^+$ is infinite, ϕ^+ is said to be <u>not integrable</u>.

The Lebesgue integral of $\int_E \phi^-$ follows analogously. Then,

$$\int_E \phi = \int_E \phi^+ - \int_E \phi^-.$$

Note that $\int_E \phi$ is well defined in that infinity is not possible (by definition) for either integral on the right hand side of the equality.

Let ϕ_1 and ϕ_2 be measurable on the measurable set E where ϕ_1 or ϕ_2 are not necessarily bounded. The following theorems are important:

(i) Let $|\phi_1(x)| \leq \phi_2(x)$ ae, where $x \in E$ and suppose ϕ_2 is integrable on E. Then ϕ_1 is integrable on E and $\int_E |\phi_1| \leq \int_E \phi_2$.

(ii) ϕ_1 is integrable on E if and only if $|\phi_1|$ is integrable on E. Hence, $|\int \phi_1| \leq \int |\phi_1|$.

(iii) Let $\int_E \phi_1$ exist (i.e., $\int_E \phi_1$ is finite). Then ϕ_1 is finite ae on E.

(iv) Let E^1 be a measurable subset of E. Then if $\int_E \phi_1$ exists, (i.e., is finite), then

$$\int_{E^1} \phi_1 \text{ exists and } \int_{E^1} |\phi_1| < \int |\phi_1|$$

(v) Theorems (ii), (iii), (iv), (v) of Section 2.4.1 (Bounded Functions) apply to unbounded functions.

EXAMPLE 2.1.

A set E in **R** is said to be denumerable if each element on E can be put into a one-to-one correspondence with the positive integers. A set which is either empty (has no elements), finite, or denumerable is said to be countable. The positive Rational numbers Q^+ where $Q^+ = \{x \in \mathbf{R}: x = \frac{m}{n}$ where m and n are positive integers and $n \neq 0\}$ are countable. To show this, arrange all the elements in Q^+ as follows:

$$Q^+ = \{0; 1; \tfrac{1}{2}; \tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{4}, -, \tfrac{3}{4}, -; \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, -; \cdots\}$$

In the above set, the rationals are arranged according to common denominators. It is noted that the blanks indicate rationals already accounted for; for example, $\frac{2}{4} = \frac{1}{2}$ and $\frac{4}{4} = \frac{1}{1} = 1$. This arrangement of Q^+ enables the positive rationals to be counted. That is, Q^+ is countable.

EXAMPLE 2.2.

Let E be a countable set of real numbers. Then the measure of E, mE, is zero. This can be shown by choosing an $\varepsilon > 0$ and enclosing each point $x_j \in E$ by open intervals of lengths smaller than or equal to $\varepsilon/2$, $\varepsilon/4$, $\varepsilon/8$, $\varepsilon/16, \cdots$. Then the exterior measure of E, $m_e(E)$, is given by

$$m_e(E) \leq \varepsilon/2 + \varepsilon/4 + \varepsilon/8 + \varepsilon/16 + \cdots = \varepsilon$$

As $\varepsilon \to 0$ (arbitrarily chosen), then $m_e(E) \to 0$. Thus, $m(E) = 0$.

EXAMPLE 2.3.

Consider the unit interval [0,1]. Construct a function ϕ on [0,1] as follows: Take a piece of paper of height 1 and width 1/2. Cut the paper into halves, always preserving the height with a length of 1. Center one of two pieces of paper in the interval [0,1] so that the subinterval [3/8, 5/8] is covered. Now cut the remaining piece of paper into quarters, resulting in 4 pieces of paper for height 1 and width 1/16. Place two of these pieces in the center of the two intervals [0, 3/8) and (5/8, 1]. Continuing this procedure, the interval [0,1] will have the function ϕ defined such that any subinterval E^1 (of any length) contained in [0,1] will have functions values of $\phi(x) = 1$ for some $x \in E^1$. In fact, most intervals $E^1 \subset [0,1]$ will have function values of $\phi(x_1) = 0$ and $\phi(x_2) = 1$ for some x_1 and $x_2 \in E^1$.

The Riemann integral applied to $\phi(x)$ an [0,1] will result in $\int_0^1 \phi(x)dx$ being undefined as the upper Riemann sum $\bar{S}$ is given by

$$\bar{S} = \sum_{j=1}^{n} (1)\ (\Delta x_j)$$

and the lower Riemann sum $\underline{S}$ contains several intervals Δx_j where $\min(\phi(x)) = 0$. Thus, $\bar{S} \neq \underline{S}$ and the Riemann integral is undefined.

The Lebesgue integral, on the other hand, solves this integration problem by noting for E = [0,1]

$$\int_E \phi = (0)m\{x \in E: \phi(x) = 0\} + (1)m\{x \in E: \phi(x) = 1\}$$

$$= m\{x \in E: \phi(x) = 1\}$$

$$= 1/2.$$

EXAMPLE 2.4.

Let ϕ be a nonnegative measurable function defined on the measurable set E. Then there exists a sequence $\langle\phi_n\rangle$ of functions measurable on E such that (i) the $\langle\phi_n\rangle$ are monotonically increasing (i.e., $\phi_{n+1}(x) \geq \phi_n(x)$ for $x \in E$ and all n); (ii) $\phi_n(x) \geq 0$; (iii) as $n \to \infty$, $\phi_n(x) \to \phi(x)$; (iv) each ϕ_n has only a finite number of values; (v) $\phi_n(x) \in \mathbf{R}$. Such a sequence of functions places confidence that ϕ can be approximated to within any accuracy desired using the maximum norm as the standard of distance.

The $<\phi_n>$ is constructed by defining sets $E_{jn} = \{x \in E: j/2^n \leq \phi(x) < (j+1)/2^n\}$ and for $j = n2^n$, $E_{jn} = \{x \in E: \phi(x) \geq n\}$. These sets have the properties that the sets E_{jn} are all mutually disjoint and yet the union of the E_{jn} equals the parent set, E. Letting $\phi_n = j/2^n$ on E_{jn} results in the desired sequence of approximation functions.

2.5. Key Theorems in Integration Theory

In using the Best Approximation Method, or any numerical method, the engineer develops a sequence of approximation functions $<\phi_n>$ by repeated tries at solving the governing operator equation. For example, ϕ_{n+1} may be nearly the same approximation as ϕ_n except that for ϕ_{n+1}, additional nodal points or collocation points were added to the computation.

Generally, the engineer evaluates his work effort by noticing how much the computational results change with the use of a more discretized model; and if the differences between the two modeling efforts are small, then the engineer concludes a good approximation has been achieved.

Mathematically, the above "convergence" criteria is defined by $||\phi_n - \phi_{n+1}||$ is small where typically the norm is the judgement of the analyst, and $||\phi_n - \phi_{n+1}||$ is evaluated at a few discrete points in the problem domain.

In this section, a few key theorems are presented which will be of importance in establishing convergence properties of a sequence of functions.

2.5.1. Monotone Convergence Theorem

Let $<\phi_n>$ be a sequence of nonnegative, monotically increasing functions measurable on the measurable set E. Suppose that as $\phi_n(x) \to \phi(x)$, as $n \to \infty$. Then as $n \to \infty$, $\int_E \phi_n = \int_E \phi$.

EXAMPLE 2.5.

Let $<\phi_n>$ be a sequence defined by

$$\phi_n(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq 1 - \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

Let $\phi(x)$ be defined by $\phi(x) = 1$ for $E = \{x: 0 \leq x \leq 1\}$. Then $<\phi_n>$ is a monotonically increasing sequence of measurable functions and as $n \to \infty$, $\phi_n(x) \to \phi(x)$ for $x \in [0,1]$. Then $\int_E \phi_n = (1 - \frac{1}{n})$. Then as $n \to \infty$, $\int_E \phi_n \to 1 = \int_E \phi$.

EXAMPLE 2.6.

Define a sequence $<\phi_n>$ by

$$\phi_n(x) = \begin{cases} \frac{1}{n}, & \text{for } 0 \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$$

and define $\phi(x) = 0$ for $x \geq 0$. Then as $n \to \infty$, $\phi_n(x) \to \phi(x)$. But $\int \phi_n = \frac{1}{n}(n) = 1$ and $\int \phi = 0$. Hence as $n \to \infty$, $\int \phi_n \neq \int \phi$, showing that the Monotone Convergence theorem only applies to a monotonically increasing sequence of measurable functions.

2.5.2. Dominated Convergence Theorem

Let $\langle\phi_n\rangle$ be a sequence of functions measurable on the measurable set E such that as $n\to\infty$, $\phi_n(x)\to\phi(x)$ ae. Let $f(x)$ be a non-negative function measurable on E such that $|\phi_n(x)| \le f(x)$ for $n = 1,2,\cdots$. Then as $n\to\infty$, $\int_E \phi_n = \int_E \phi$.

2.5.3. Egorov's Theorem

Let $\langle\phi_n\rangle$ be a sequence of functions measurable on a set E such that as $n\to\infty$, $\phi_n(x)\to\phi(x)$ ae on E where $\phi(x)$ is bounded. Then for any real number $\delta>0$ there exists a set E^1 in E such that $mE^1 > mE - \delta$ and as $n\to\infty$, $\phi_n(x)\to\phi(x)$ uniformly.

2.6 L_p Spaces

The set of all functions $\phi(x)$ defined on E such that the function $|\phi(x)|^p$ for $p \ge 1$ is Lebesgue integrable is denoted by L (E). That is, a function $\phi(x)$ is an element of $L_p(E)$ if and only if

$$\int_E |\phi|^p < \infty.$$

EXAMPLE 2.7.

Let $p = 2$ in the above description of L_p spaces. Then a function ϕ is in $L_2(E)$ if and only if $\int_E \phi^2 < \infty$.

2.6.1. m-Equivalent Functions

Let ϕ_1 and ϕ_2 be functions measurable on the measurable set E. Then if $\phi_1 = \phi_2$ ae on E, ϕ_1 and ϕ_2 are said to be m-equivalent. That is, $m\{x \in E: \phi_1(x) \neq \phi_2(x)\} = 0$. Because many functions can be m-equivalent, the notation $[\phi]$ is used to designate the equivalence class of functions which are m-equivalent to ϕ on E.

The Lebesque space $L_1(E)$ with the usual Lebesgue measure (m) consists of elements which are equivalence classes. In $L_1(E)$, the norm is defined for $[\phi]$ by $||[\phi]||_1 = \int_E |\phi|$. Because integration on sets of measure zero is zero, the equivalence class notation $[\phi]$ can be dropped in all future work. However, the reader must recall that all measurable functions being studied are but single elements of an equivalence class of measurable functions.

2.6.2. The Space L_p

Let $1 \leq p < \infty$. Then the space $L_p(E)$ consists of all equivalence classes of functions measurable on the measurable set E. The norm is defined for $\phi \in L_p(E)$ by $||\phi||_p = [\int_E |\phi|^p]^{\frac{1}{p}}$. Then L_p is a normed linear space whose norm definition guarantees that every Cauchy sequence $\langle \phi_n \rangle$ of elements $\phi_n \in L_p(E)$ converges to a limit $\phi \in L_p(E)$. Thus $L_p(E)$ is a Banach space.

2.6.3. Hölder's Inequality

Let $\phi_p \in L_p(E)$ and $\phi_q \in L_q(E)$ where $1/p + 1/q = 1$ and $p > 1$. Then (i) the product $\phi_p\phi_q \in L_1(E)$, and (ii) $||\phi_p\phi_q||_1 \leq ||\phi_p||_p ||\phi_q||_q$; that is, $\left|\int_E \phi_p\phi_q\right| \leq \left|\int_E |\phi_p|^p\right|^{1/p} \left|\int_E |\phi_q|^q\right|^{1/q}$

2.6.4. Cauchy-Bunyakovskii-Schwarz Inequality

Let $p = 2$ in Hölder's Inequality. Then $q = 2$ and

$$\left|\int_E \phi_1\phi_2\right| \leq \int_E |\phi_1\phi_2| < ||\phi_1||_2 ||\phi_2||_2$$

where ϕ_1 and ϕ_2 are in $L_2(E)$.

2.6.5. Minkowski's Inequality

Let ϕ_1 and ϕ_2 be in $L_p(E)$ for $p \geq 1$. Then $\phi_1 + \phi_2$ are in $L_p(E)$ and

$$||\phi_1 + \phi_2||_p \leq ||\phi_1||_p + ||\phi_2||_p$$

2.6.6. Triangle Inequality

Let $p = 2$ in the Minkowski Inequality. Then

$$||\phi_1 + \phi_2||_2 \leq ||\phi_1||_2 + ||\phi_2||_2$$

That is,

$$\int_E (\phi_1 + \phi_2)^2 \leq \int_E \phi_1^2 + \int_E \phi_2^2$$

2.7. The Metric Space, L_p

Given a space $L_p(E)$, the distance between two vectors (points, or elements) in $L_p(E)$ is defined by

$$D(\phi_1\ \phi_2) = ||\phi_1 - \phi_2||_p = [\int_E |\phi_1 - \phi_2|^p]^{1/p}$$

for ϕ_1 and ϕ_2 in $L_p(E)$. Then it is seen that (i) $D(\phi_1,\phi_2) \geq 0$; (ii) $D(\phi_1,\phi_2) = 0$ implies $\phi_1 = \phi_2$ ae on E (i.e., ϕ_1 and ϕ_2 are in the same equivalence class), (iii) $D(\phi_1,\phi_2) = D(\phi_2,\phi_1)$, and (iv) from Minkowski's inequality $D(\phi_1,\phi_2) \leq D(\phi_1,\phi_3) + D(\phi_3,\phi_2)$ where ϕ_3 is also in $L_p(E)$.

2.8. Convergence of Sequences

We are now prepared to discuss how well our approximation function ϕ_n approximates the true solution ϕ of an operator equation. As we develop more and more "accurate" approximations, we are generating a sequence of approximations $<\phi_n>$ which, as $n\to\infty$, we hope that $\phi_n\to\phi$.

The first question to be asked is the character of the working space, and then the norm needs to be evaluated. Because there are several modes of convergence, the type of convergence to be studied needs to be determined.

2.8.1. Common Modes of Convergence

A sequence $<\phi_n>\to\phi$ uniformly if for every $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $n \geq N(\varepsilon)$ implies $|\phi_n(x) - \phi(x)| < \varepsilon$ for $x \in E$. This type of convergence is denoted by U.

A sequence $\langle\phi_n\rangle \to \phi$ pointwise if for every $\varepsilon > 0$ and $x \in E$ there exists an $N(\varepsilon, x)$ such that $n \geq N(\varepsilon, x)$ implies $|\phi_n(x) - \phi(x)| < \varepsilon$. This type of convergence is denoted by P.

Similar to pointwise convergence, ae convergence indicates pointwise convergence except for a set $E^1 \subset E$ where $m(E^1) = 0$. This convergence is denoted by AE.

From the above definitions, $U \to P \to AE$, but no other conclusions can be drawn.

2.8.2. Convergence in L_p

$\langle\phi_n\rangle \to \phi$ in L_p if for every $\varepsilon > 0$ there exists a $N(\varepsilon)$ such that if m and $n \geq N(\varepsilon)$, then

$$||\phi_m - \phi_n||_p = \left[\int_E |\phi_m - \phi_n|^p\right]^{1/p} < \varepsilon$$

2.8.3. Convergence in Measure (M)

A sequence of measurable functions $\langle\phi_n\rangle \to \phi$ in measure if as $n \to \infty$, and

$$m\{x \in E: |\phi_n(x) - \phi(x)| \geq \lambda\} = 0$$

for any $\lambda \in \mathbf{R}$.

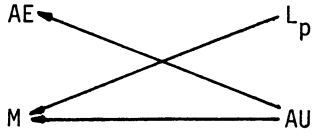
2.8.4. Almost Uniform Convergence (AU)

Let $\langle\phi_n\rangle$ and ϕ be functions measurable on the measurable set E. Then $\langle\phi_n\rangle \to \phi$ AU if for any $\delta > 0$ there exists a set $E_\delta \subset E$ such that $\langle\phi_n\rangle \to \phi$ U on the set $E^1 = E - E_\delta$, and $m(E_\delta) < \delta$.

2.8.5. What Implies What?

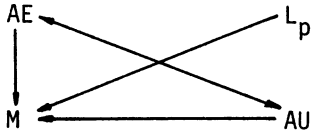
Each type of convergence may or may not imply another type of convergence. A convenient summary of the cross-implications in the types of convergence can be made in diagram form as follows:

For any measurable set E,



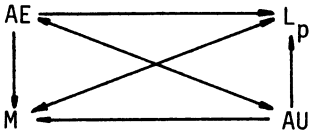
(3 implications)

If mE is finite, two more implications can be concluded,



(5 implications)

Notice in this case that AE only implies AU and M and not L_p. Also, L_p only implies M convergence. Should $\langle\phi_n\rangle$ be a sequence of functions such that $|\phi_n(x)| \leq f(x)$ for all $x \in E$ and $f(x)$ is integrable on E, then three more implications are added,



(8 implications)

Here, AU, AE and M convergence implies L_p convergence, yet L_p convergence only implies convergence in M. This should be remembered by the analyst as the Best Approximation Method solves for the best approximation in the L_p sense (where $p = 2$).

2.8.6. Counterexamples

It is very important to understand the limitations between the various types of convergence. To demonstrate the failures of one type of convergence implying another type of convergence, several classic counterexamples are provided in the following examples.

EXAMPLE 2.8.

Let $\phi_n(x) = \begin{cases} n^{-1/p}, & \text{for } 0 \le x \le n \\ 0, & \text{for } x \ge 0 \end{cases}$

Then as $n \to \infty$, $\phi_n(x) \to \phi(x)$ uniformly (or U) where $\phi(x) = 0$ for $x \ge 0$. But

$$||\phi_n - \phi||_p = \left(\int_0^{\infty} |\phi_n - \phi|^p\right)^{1/p} = \left(\int_0^n (n^{-1/p})^p\right)^{1/p} = \left(\int_0^n 1/n\right)^{1/p};$$

and as $n \to \infty$, $||\phi_n - \phi||_p \to 1$. Therefore, U does not imply L_p.

EXAMPLE 2.9.

Let $\phi_n(x) = \begin{cases} n^2, & \text{for } 1/n \le x \le 2/n \\ 0, & \text{for } x \ge 0 \end{cases}$

Then as $n \to \infty$, $\phi_n(x) \to \phi(x) = 0$ pointwise,(P), for $x \ge 0$. But

$$||\phi_n - \phi||_p = \left(\int_{1/n}^{2/n} (n^2)^p\right)^{1/p} = \left(\int_{1/n}^{2/n} n^{2p}\right)^{1/p} = (n^{2p-1})^{\frac{1}{p}} = n^{2-1/p}$$

Thus as $n \to \infty$, $||\phi_n - \phi||_p \to \infty$, Therefore, P does not imply L_p.

EXAMPLE 2.10.

Let $\phi_n(x)$ be the triangle such that $\phi_n(x = 0) = 0$, $\phi_n(x = \frac{1}{n}) = n^2$, $\phi_n(x = \frac{2}{n}) = 0$, and $\phi_n(x > 2/n) = 0$, for all n

and $x \geq 0$. Let $\phi(x) = 0$ for $x \geq 0$. Then as $n \to \infty$, $\phi_n(x) \to \phi(x)$ pointwise (P). But $||\phi_n - \phi||_1 = (\int_0^{2/n} \phi_n) = n$. Thus as $n \to \infty$, $||\phi_n - \phi||_1 \to \infty$ showing that P does not imply L_p.

EXAMPLE 2.11.

Let $\phi_n(x)$ be defined on $E = \{x: x \geq 0\}$ by

$$\phi_n(x) = \begin{cases} 0, \text{ for } 0 \leq x \leq n \\ \text{triangle, where } \phi_n(x=n) = 0, \\ \quad \phi_n(x=n+1) = 1, \ \phi_n(x=n+2) = 0 \\ 0, \text{ for } x \geq n+2 \end{cases}$$

Let $\phi(x) = 0$ for $x \geq 0$. Then for $\alpha > 0$ and any n, $m\{x \in E: |\phi_n(x) - \phi(x)| \geq \alpha\} = m\{x \in E: \phi_n(x) \geq \alpha\} = 2$; that is, the length of the interval $(n, n+2)$. Thus as $n \to \infty$ $m\{x \in E: |\phi_n(x) - \phi(x)| \geq \alpha\} = 2$. Therefore, $\phi_n(x) \to \phi(x)$ pointwise but $\phi_n(x)$ does not $\to \phi(x)$ in measure. Hence, P does not imply M.

EXAMPLE 2.12.

From EXAMPLE 2.9, ϕ_n does not converge in L_p to ϕ on $E = \{x: x \geq 0\}$. But $m\{x \in E: |\phi_n(x) - \phi(x)| \geq \alpha\} \to 1/n$ as $\alpha \to 0$. And as $n \to \infty$, $m\{x \in E: |\phi_n(x) - \phi(x)| \geq \alpha\} \to 0$. Thus, $\phi_n(x) \to \phi(x)$ in measure (M), but not in L_p.

EXAMPLE. 2.13.

Construct the sequence $\langle \phi_n \rangle$ such that each ϕ_n is a rectangle of heigth 1 and a decreasing base according to the sequence $1, 1/2, 1/2, 1/3, 1/3, 1/3, \cdots$. Each rectangle is placed on $E = [0,1]$ so that ϕ_1 covers $[0,1]$, ϕ_2 covers $[0, 1/2]$, ϕ_3 covers $[1/2, 1]$, ϕ_4 covers $[0, 1/3]$, ϕ_5 covers $[1/3, 2/3]$, ϕ_6 covers

[2/3, 1], and so forth. Let $\phi(x) = 0$ on E. As $n \to \infty$, the $m\{x \in E: |\phi_n(x) - \phi(x)| \geq \alpha\} \to 0$. Also, as $n \to \infty$

$||\phi_n - \phi||_p = (\int_E \phi_n^p)^{\frac{1}{p}} = (\frac{1}{n})^{1/p} \to 0$. But for any $x \in E$, consider the sequence of values $\langle\phi_n(x)\rangle$ which is composed of zeroes and ones. That is, as $n \to \infty$, each "cycle" of ϕ_n as the rectangle bases shift from covering $x = 0$ to finally covering the point $x = 1$, will cover any point $x \in E$. Hence $\phi_n(x)$ does not pointwise converge anywhere. Thus, $\phi_n \to \phi$ in both M and L_p, and yet ϕ_n does not converge to ϕ pointwise, even on a finite measure space.

2.9. Capsulation

This chapter reviewed the basic theory of Lebesque integration needed to develop the generalized Fourier series theory, and developed the theory fundamental to converging sequences. Because the engineer is developing a sequence of approximations $\langle\phi_n\rangle$ to ϕ on E, he needs to know that if given enough computer power, will $\phi_n \to \phi$ as $n \to \infty$ and if so, by what standard of convergence.

The Best Approximation Method deals with L_2 convergence and, from section 2.8, it is seen that L_2 convergence only guarantees convergence in measure. But in engineering problems in general, additional hypothesis are available such as continuity, piecewise continuity, and so forth. As a result of these additional hypothesis, the Best Approximation Method provides additional implications than that shown in the logic diagrams of section 2.8.5.

These additional hypothesis will be discussed in Chapter 3 along with the Hilbert space environment, generalized Fourier series, and finite dimensional vector space representations of piecewise continuous functions defined over the problem domain.

CHAPTER 3

HILBERT SPACE AND GENERALIZED FOURIER SERIES

3.0. Introduction

The subjects of inner product, Hilbert space, generalized Fourier series, and vector space representations are all used in the Best Approximation Method. To introduce these concepts let the underlying Banach space be $\mathbf{R}^3$ where each vector (or element) $\xi \in \mathbf{R}^3$ is of the form $\xi = (x,y,z)$. Let ξ_1 and ξ_2 be two distinct vectors and $\overrightarrow{0\xi}$ be the straight line through points $(0,0,0)$ and $\xi_1 = (x_1, y_1, z_1)$. The task is to find the point $\xi^* = (x^*, y^*, z^*)$ in $\overrightarrow{0\xi}$, which is closest to the point $\xi_2 = (x_2, y_2, z_2)$. Closest is defined to mean the minimum value of the norm used in $\mathbf{R}^3$. The norm used is the l_2 norm (Euclidean norm) $||\xi||_2 = ||(x,y,z) - (0,0,0)||_2 = [x^2 + y^2 + z^2]^{1/2}$. Any point ξ in $\overrightarrow{0\xi}$, has a distance from ξ_2 calculated from

$D^2(\xi, \xi_2) = ||\xi, \xi_2||^2 = (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2$. But ξ in $\overrightarrow{0\xi}$, can be written as $\xi = \lambda\xi_1$ where $\lambda \in \mathbf{R}$. Thus

$$D^2(\xi, \xi_2) = (\lambda x_1 - x_2)^2 + (\lambda y_1 - y_2)^2 + (\lambda z_1 - z_2)^2$$

$$= \lambda^2 (x_1^2 + y_1^2 + z_1^2) - 2\lambda(x_1x_2 + y_1y_2 + z_1z_2) + (x_2^2 + y_2^2 + z_2^2)$$

using vector dot product notation where

$\xi_1 \cdot \xi_2 = x_1x_2 + y_1y_2 + z_1z_2$, $D^2(\xi, \xi_2) = \lambda^2 \xi_1 \cdot \xi_1 - 2\lambda\xi_1 \cdot \xi_2 + \xi_2 \cdot \xi_2$.

The above result can be obtained more quickly from the definition of the norm by noting $D^2(\xi,\xi_2) = ||\lambda\xi_1 - \xi_2||^2$ $= (\lambda\xi_1 - \xi_2)\cdot(\lambda\xi_1 - \xi_2) = \lambda^2\xi_1\cdot\xi_1 - 2\lambda\xi_1\cdot\xi_2 + \xi_2\cdot\xi_2$. Differentiating the equation with respect to λ gives that value of $\lambda \in \mathbf{R}$ which minimizes $D^2(\xi,\xi_2)$; that is, $\lambda = (\xi_1\cdot\xi_2)/(\xi_1\cdot\xi_1)$. Thus, the point ξ in $\overrightarrow{0\xi}$, must be given by $\xi^* = \xi_1(\xi_1\cdot\xi_2)/(\xi_1\cdot\xi_1)$.

Because ξ^* is closest to point ξ_2, the line $\overrightarrow{\xi^*\xi_2}$ must be orthogonal to the line $\overrightarrow{0\xi_2}$. This is described by the dot product $\overrightarrow{\xi^*\xi_2}\cdot\overrightarrow{0\xi_1} = 0$. That is, for $\overrightarrow{\xi^*\xi_2} = (\xi_2 - \lambda\xi_1)$ and $\overrightarrow{0\xi_1} = \xi_1$, $0 = (\xi_2 - \lambda\xi_1)\cdot\xi_1 = \xi_2\cdot\xi_1 - \lambda\xi_1\cdot\xi_1$ giving $\lambda = \xi_2\cdot\xi_1/\xi_1\cdot\xi_1$.

This geometric interpretation of closeness extends immediately to vector spaces $\mathbf{R}^n$ for all $n \geq 1$, and for the l_2 norm, even though there fails to be a visual representation expedient to represent functions on [a,b] as vectors of dimension n in order to estimate the generalized Fourier coefficients. For example, let $E = \{x: 0 \leq x \leq 1\}$ and subdivide E = [0,1] with 11 collocation points located at coordinates 0, 1/10, 2/10,$\cdots$, 9/10, 1. Let f be a function defined on [0,1]. Then the vector representation **F** of f in R" is given by **F** = (f(0), f(1/10),$\cdots$,f(1)). As n gets large, **F** represents f(x) on E more accurately. Obviously, n may have to be large in order to capture in **F** all the characteristics of f(x).

3.1. Inner Product and Hilbert Space (Finite Dimension Spaces)

Definition 3.1: (Inner Product)

Let S be a real linear space. For any vectors ξ_1, ξ_2, and ξ_3 in S define a real number denoted by (ξ_1, ξ_2) which satisfies

(i) $(\xi_1, \xi_1) \geq 0$

(ii) $(\xi_1, \xi_1) = 0$ implies $\xi_1 = \mathbf{0} \in S$

(iii) $(\lambda\xi_1, \xi_2) = \lambda(\xi_1, \xi_2)$ for $\lambda \in \mathbf{R}$

(iv) $(\xi_1, \xi_2) = (\xi_2, \xi_1)$

(v) $(\xi_1 + \xi_2, \xi_3) = (\xi_1, \xi_3) + (\xi_2, \xi_3)$

Then (,) is called an inner product

Definition 3.2: (Hilbert Space)

A linear space S on which an inner product is defined is called an inner product space. If S is also complete, then S is called a Hilbert space and the norm is defined by $||\xi||^2 = (\xi, \xi)$ for $\xi \in S$.

Several properties are associated to an inner product space S.

(i) Let ξ_1 and ξ_2 be in S. Then $|(\xi_1, \xi_2)| \leq ||\xi_1||\ ||\xi_2||$. Note that the norm follows from Definition 3.2.

(ii) Let ξ_1 and ξ_2 be in S. Then the angle θ between ξ_1 and ξ_2 is given by

$$\theta = \cos^{-1}(\xi_1, \xi_2)/(||\xi_1||\ ||\xi_2||)$$

(iii) Elements ξ_1 and ξ_2 in S are orthogonal if and only if $(\xi_1, \xi_2) = 0$. That is, $\theta = \pi/2$.

(iv) Similar to vector space $\mathbf{R}^n$, the orthogonal projection of ξ_1 onto ξ_2 is given by $\xi_1(\xi_1, \xi_2)/(\xi_1, \xi_1)$.

(v) A weighted inner product is an inner and is product is defined in $\mathbf{R}^n$ by $(\xi_1, \xi_2) = w_1 x_1 x_2 + w_2 y_1 y_2 + \cdots$, where the w_i are positive numbers.

3.2. Infinite Dimension Spaces

In the vector representation **F**, as $n \to \infty$ the Dim $(\mathbf{F}) \to \infty$. However, the above discussed theory still applies in infinite dimension spaces as in finite dimension spaces. For our purposes, we are especially interested in the L_2 space.

3.2.1. L_2 Space

The space of all real valued functions such that $||f||^2 = \int_E f^2 < \infty$ is call $L_2(E)$.

3.2.2. Inner Product in $L_2(E)$

Let ϕ_1 and ϕ_2 be in $L_2(E)$. The inner product is defined by $(\phi_1,\phi_2) = \int_E \phi_1\phi_2$.

3.2.3. Orthogonal Functions

Let $E = (a,b)$. Let ϕ_1 and ϕ_2 be in $L_2(E)$. Then ϕ_1 and ϕ_2 are orthogonal in E if $(\phi_1,\phi_2) = \int_a^b \phi_1(x)\phi_2(x)dx = 0$. A set of functions $\phi_1,\phi_2,\cdots$ in $L_2(E)$ is said to be an orthogonal set in E if $(\phi_i,\phi_j) = 0$ for $i \neq j$.

EXAMPLE 3.1.

Perhaps the most classic example of orthogonal functions are the functions sinmx for $m = 1,2,\cdots$ and $E = (-\pi,\pi)$. Then $\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0$ for all $m \neq n$.

3.2.4. Orthonormal Functions

It is desirable to have the property in an orthogonal set of functions $\{\phi_n\}$ that $(\phi_n,\phi_n) = 1$. Let $\{f_n\}$ be an orthogonal set of functions in E and define functions $\phi_n(x) = f_n(x)/||f_n||$. Then $\{\phi_n\}$ is said to be an orthonormal set of functions in E such that $(\phi_i,\phi_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

EXAMPLE 3.2.

The set of functions $\{\sin mx/\sqrt{\pi}\}$ is orthonormal in $(-\pi,\pi)$.

3.3. Approximations in $L_2(E)$

Let $\{\phi_n\}$ be an orthonormal set of functions in E and $\phi \in L_2(E)$. Define an approximation function by the series $\gamma_1\phi_1 + \gamma_2\phi_2 + \cdots$. Compute constants γ_j, by $\gamma_j = \int_E \phi\ \phi_j$.

Then the γ_j are called the generalized Fourier coefficients. Because both ϕ and $\{\phi_n\}$ are in $L_2(E)$, the γ_j values exist.

It is noted that as additional functions are added to a finite set of basis functions $\{\phi_n\}$, the generalized Fourier coefficients γ_j are still calculated by the integral $\int_E \phi\ \phi_j$, and the mode of convergence being considered is L_p for $p = 2$.

3.3.1. Parseval's Identity

Let $\{\phi_n\}$ be an orthonormal set of functions in E with generalized Fourier coefficients computed by $\lambda_j = \int_E \phi\ \phi_j$. Also let ϕ and $\{\phi_n\}$ be in $L_2(E)$. Then if $||(\lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_m\phi_m) - \phi|| \to 0$ as $n \to \infty$, we have L_p convergence with $p = 2$ and $\int_E \phi^2 = \sum_{j=1}^{\infty} \lambda_j{}^2$. This equality is called Parseval's Identity.

3.3.2. Bessel's Inequality

From Parseval's identity, it can be shown that $\int_E \phi^2 \geq \sum_{j=1}^{\infty} \lambda_j^2$ which is called Bessel's Inequality. This inequality is used in the Best Approximation Method to evaluate the success and rate of L_2 convergence.

3.4. Vector Space Representation for Approximations: An Application

Consider the function $f(x) = e^x$ on $E = [0,1]$.

The approximation problem is to find the best approximation of $f(x) = e^x$ using a combination of the basis functions $\{f_n\} = \{1, x, x^2\}$.

Let $\{f_1, f_2, f_3\} = \{1, x, x^2\}$. Then vectors $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$ can be determined by evaluating each f_i at, for example, points $x = 0, 0.25, 0.50, 0.75, 1.0$, respectively. Then

$$\mathbf{F}_1 = (1,1,1,1,1)$$

$$\mathbf{F}_2 = (0, 0.25, 0.50, 0.75, 1.0)$$

$$\mathbf{F}_3 = (0, 1/16, 1/4, 9/16, 1.0)$$

and

$$\mathbf{F} = (1, 1.284, 1.649, 2.117, 2.718)$$

where $\mathbf{F}$ is $\phi = e^x$ evaluated at the given points in E. The set $\{\mathbf{F}_j\}$ is orthonormalized by the Gramm-Schmidt technique as follows:

$$\mathbf{G}_1 = \mathbf{F}_1 / ||\mathbf{F}_1|| = (0.447, 0.447, 0.447, 0.447, 0.447)$$

$$\mathbf{G}_2 = (\mathbf{F}_2 - (\mathbf{F}_2, \mathbf{G}_1)\mathbf{G}_1) / ||\mathbf{F}_2 - (\mathbf{F}_2, \mathbf{G}_1)\mathbf{G}_1||$$

$$= (-0.633, -0.316, 0, 0.316, 0.633)$$

and

$$\mathbf{G}_3 = (\mathbf{F}_3 - (\mathbf{F}_3, \mathbf{G}_1)\mathbf{G}_1 - (\mathbf{F}_3, \mathbf{G}_2)\mathbf{G}_2)/$$

$$||\mathbf{F}_3 - (\mathbf{F}_3, \mathbf{G}_1)\mathbf{G}_1 - (\mathbf{F}_3, \mathbf{G}_2)\mathbf{G}_2||$$

$$= (0.534, -0.265, -0.534, -0.265, 0.534)$$

It is verified that $(\mathbf{G}_i, \mathbf{G}_j) = 1$ for $i = j$ and $(\mathbf{G}_i, \mathbf{G}_j) = 0$ for $i \neq j$.

Then the generalized Fourier coefficients γ_j are given by

$$\gamma_1 = (\mathbf{G}_1, \mathbf{F}) = 3.919$$

$$\gamma_2 = (\mathbf{G}_2, \mathbf{F}) = 1.351$$

$$\gamma_3 = (\mathbf{G}_3, \mathbf{F}) = 0.203$$

Thus the best approximation of $\mathbf{F}$ is given by $\gamma_1\mathbf{G}_1 + \gamma_2\mathbf{G}_2 + \gamma_3\mathbf{G}_3$.

The vector representations of the $\{f_j\}$ are used to develop estimates of γ_j for use with f_j, $j = 1,2,3$. Thus by the back substitution through the above Gramm-Schmidt procedure it is seen that the $\{\mathbf{G}_j\}$ are the vector equivalents for the $\{f_j\}$ modified into $\{g_j\}$ defined on $[0,1]$ by

$$g_1 = 0.447$$

$$g_2 = 1.265x - 0.632$$

$$g_3 = 4.274x^2 - 4.274x + 0.534$$

Using the γ_j from the dimension 5 vector representations gives the best approximation estimate of

$$\hat{f}(x) = \gamma_1 g_1 + \gamma_2 g_2 + \gamma_3 g_3 = 0.868x^2 + 0.841x + 1.$$

A comparison between $\hat{f}(x)$ and $f(x)$ is as follows:

Table 3.1 Approximation Results of e^x by the Best Approximation Method

x	$f(x) = e^x$	$\hat{f}(x)$	$f(x) - \hat{f}(x)$
0	1.000	1.000	0.000
0.125	1.1331	1.1187	0.014
0.250	1.2840	1.2645	0.020
0.375	1.4550	1.4374	0.018
0.500	1.6487	1.6375	0.011
0.625	1.8682	1.8647	0.0035
0.750	2.1170	2.119	-0.0020
0.875	2.3989	2.400	-0.0011
1.000	2.7183	2.709	0.0093

From Table 3.1, the tabled maximum relative error occurs at $x = 0.25$ where the relative error is 1.56 percent.

There are two points to consider: (i) the approximation can be improved by increasing the dimension of the vector representation; however, there is a limit to how well the $\{1,x,x^2\}$ functions can approximate e^x on $E = [0,1]$; and (ii) by increasing the set of basis functions, the approximation can be improved. Both of these two concepts are utilized in the Best Approximation Method. The first error is typically called an integration error, whereas the second error is labeled an error due to an insufficient number of basis functions.

CHAPTER 4

LINEAR OPERATORS

4.0. Introduction

Many of the mathematical relationships used in engineering analysis fall into the category of being linear operator equations. In this chapter, background theory is presented in the characteristics of linear operators. Additionally, several of the more common linear operators which are of interest in engineering studies are discussed.

4.1. The Derivative as a Linear Operator

Consider the derivative operation d( ) on two differentiable functions f and g. We know that $d(f+g) = d(f) + d(g)$, and $d(\lambda f) = \lambda d(f)$ for λ a constant number. This pair of relationships indicate that d( ) is a linear operator.

4.2. Linear Operators

Using Section 4.1 as a guideline, the definition of a Linear Operator is as follows:

Definition 4.1: (Linear Operator)

Let S and T be two linear spaces, and suppose L (or L( )) is a function such that $L(s) \in T$ for all $s \in S$. Then L is a linear operator if for all $s_1, s_2 \in S$ and for any $\lambda \in \mathbf{R}$ we have $L(s_1 + s_2) = Ls_1 + Ls_2$ and $L(\lambda s_1) = \lambda L(s_1)$.

4.3. Examples of Linear Operators in Engineering

Several important mathematical relationships used in engineering studies are linear operator equations.

EXAMPLE 4.1: (Laplace Equation)

Steady state heat transfer and groundwater flow are described by the Laplace equation. Let L be the well-known Laplace equation in one-dimension, $L(\phi) = \frac{d^2\phi}{dx^2}$. To show L is a linear operator:

$$L(\phi_1 + \phi_2) = \frac{d^2}{dx^2}(\phi_1 + \phi_2) = \frac{d^2(\phi_1)}{dx^2} + \frac{d^2(\phi_2)}{dx^2} = L(\phi_1) + L(\phi_2)$$

$$L(\lambda\phi_1) = \frac{d^2}{dx^2}(\lambda\phi_1) = \frac{\lambda d^2(\phi_1)}{dx^2} = \lambda L(\phi_1)$$

Now consider L to be the Laplace equation in two-dimensions, $L(\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$:

$$L(\phi_1 + \phi_2) = \frac{\partial^2(\phi_1 + \phi_2)}{\partial x^2} + \frac{\partial^2(\phi_1 + \phi_2)}{\partial y^2}$$

$$= \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} + \frac{\partial^2\phi_2}{\partial y^2} = L(\phi_1) + L(\phi_2)$$

$$L(\lambda\phi_1) = \frac{\partial^2(\lambda\phi_1)}{\partial x^2} + \frac{\partial^2(\lambda\phi_2)}{\partial y^2} = \frac{\lambda\partial^2\phi_1}{\partial x^2} + \frac{\lambda\partial^2\phi_2}{\partial x^2} = \lambda L(\phi_1).$$

EXAMPLE 4.2: (Diffusion Equation)

Diffusion processes are common in engineering analysis.

Let $L(\phi) = \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial t}$. Then

$$L(\phi_1 + \phi_2) = \frac{\partial^2 (\phi_1 + \phi_2)}{\partial x^2} - \frac{\partial (\phi_1 + \phi_2)}{\partial t} = \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} - \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t}$$

$$= L(\phi_1) + L(\phi_2)$$

$$L(\lambda \phi_1) = \frac{\partial^2 (\lambda \phi_1)}{\partial x^2} - \frac{\partial (\lambda \phi_1)}{\partial t} = \frac{\lambda \partial^2 (\phi_1)}{\partial x^2} - \frac{\lambda \partial (\phi_1)}{\partial t} = \lambda L(\phi_1)$$

EXAMPLE 4.3: (Diffusion Equation in a Nonhomogeneous Domain)

Nonhomogenuity is the general situation in studying diffusion.

Let $L(\phi) = \frac{\partial}{\partial x} K \frac{\partial \phi}{\partial x} - C \frac{\partial \phi}{\partial t}$ where $K = K(x)$ and $C = C(x)$. Then

$$L(\phi_1 + \phi_2) = \frac{\partial}{\partial x} K \frac{\partial (\phi_1 + \phi_2)}{\partial x} - C \frac{\partial (\phi_1 + \phi_2)}{\partial t}$$

$$= \frac{\partial K}{\partial x} \frac{\partial (\phi_1 + \phi_2)}{\partial x} + K \frac{\partial^2}{\partial x^2} (\phi_1 + \phi_2) - C \frac{\partial \phi_1}{\partial t} - C \frac{\partial \phi_2}{\partial t}$$

$$= K \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial K}{\partial x} \frac{\partial \phi_1}{\partial x} - C \frac{\partial \phi_1}{\partial t} + K \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial K}{\partial x} \frac{\partial \phi_2}{\partial x} - C \frac{\partial \phi_2}{\partial t}$$

$$= L(\phi_1) + L(\phi_2)$$

$$L(\lambda\phi_1) = \frac{\partial}{\partial x} K \frac{\partial(\lambda\phi_1)}{\partial x} - C \frac{\partial(\lambda\phi_1)}{\partial t}$$

$$= \frac{\partial K}{\partial x} \frac{\partial(\lambda\phi_1)}{\partial x} + K \frac{\partial^2(\lambda\phi_1)}{\partial x^2} - C\lambda \frac{\partial\phi_1}{\partial t}$$

$$= \lambda K \frac{\partial^2(\phi_1)}{\partial x^2} + \lambda \frac{\partial K}{\partial x} \frac{\partial(\phi_1)}{\partial x} - \lambda C \frac{\partial\phi_1}{\partial t}$$

$$= \lambda L(\phi_1)$$

EXAMPLE 4.4: (An Integral Equation)

Consider the integral equation $g(x) = \int_0^x f(t)dt$.

Then $L(\phi) = \int_0^x \phi(t)dt$ is a linear operator:

$$L(\phi_1 + \phi_2) = \int_0^x [\phi_1(t) + \phi_2(t)]dt = \int_0^x \phi_1(t)dt + \int_0^x \phi_2(t)dt$$

$$= L(\phi_1) + L(\phi_2), \text{ and}$$

$$L(\lambda\phi) = \int_0^x \lambda\phi(t)dt = \lambda \int_0^x \phi(t)dt = \lambda L(\phi).$$

EXAMPLE 4.5: (A Voltera Integral Equation)

In surface runoff hydrology, the Voltera integral is used in unit hydrograph theory. Let the flow rate $q(t)$ be given by $q(t) = \int_0^t e(t-x)\phi(x)dx$ where $e(t)$ is the effective rainfall at time t, and $\phi(x)$ is the unit hydrgraph. Let $L(\phi) = \int_0^t e(t-x)\phi(x)dx$.

Then $L(\phi_1 + \phi_2) = \int_0^t e(t-x)[\phi_1(x) + \phi_2(x)]dx = L(\phi_1) + L(\phi_2)$.

Also, $L(\lambda\phi) = \int_0^t e(t-x)\lambda\phi(x)dx = \lambda L(\phi)$.

EXAMPLE 4.6: (Matrix Equations)

A typical problem in engineering computer analysis is the solution of the matrix system $\underset{\sim}{A} = \mathbf{L}\underset{\sim}{B}$ where $\underset{\sim}{A}$ is an m×1 vector of unknowns, $\underset{\sim}{B}$ is an m×1 vector of constant values, and $\mathbf{L}$ is an m×m matrix.

To show $\mathbf{L}$ is a linear operator let $\underset{\sim}{\phi}_1$ and $\underset{\sim}{\phi}_2$ be (m×1) vectors; then immediately $\mathbf{L}(\underset{\sim}{\phi}_1 + \underset{\sim}{\phi}_2) = \mathbf{L}(\underset{\sim}{\phi}_1) + \mathbf{L}(\underset{\sim}{\phi}_2)$, and $\mathbf{L}(\lambda\underset{\sim}{\phi}_1) = \lambda\mathbf{L}(\underset{\sim}{\phi}_1)$. We see that $\mathbf{L}$ is a linear operator which operates on elements of S = {(m×1) vectors of real constants}, and for all $s \in S$, $\mathbf{L}(s) \in T$. But each $t \in T$ is an (m×1) vector of real constants. Then $t \in S$, and T and S coincide. Thus, $\mathbf{L}$ is an operator from a linear space onto itself.

EXAMPLE 4.7.

In EXAMPLE 4.6, the output space T coincided with the input space S. Consider the linear operator which maps $S \to T$ where $\phi(x)$ is an element of C[a,b] and $f(x) = \int_a^x \phi(t)dt$, $x \leq b$.

Then $L(\phi) = \int_a^x \phi(t)dt$ and L maps $C[a,b] \to C[a,b]$; that is, S = T. However if $y = \int_a^b \phi(t)dt$, then $L(\phi) = \int_a^b \phi(t)dt$ is a linear operator which maps $C[a,b] \to \mathbf{R}$; that is $S \neq T$.

EXAMPLE 4.8: (A Nonlinear Operator)

Let $L(\phi) = \phi \frac{d\phi}{dx}$. Then L is a nonlinear operator as shown by noting $L(\phi_1 + \phi_2) = (\phi_1 + \phi_2) \frac{d}{dx} (\phi_1 + \phi_2)$

$$= \left(\phi_1 \frac{d\phi_1}{dx} + \phi_2 \frac{d\phi_2}{dx}\right) + \left(\phi_1 \frac{d\phi_2}{dx} + \phi_2 \frac{d\phi_1}{dx}\right) \neq \phi_1 \frac{d\phi_1}{dx} + \phi_2 \frac{d\phi_2}{dx}$$

$= L(\phi_1) + L(\phi_2)$. Thus, L is nonlinear as $L(\phi_1 + \phi_2) \neq L(\phi_1) + L(\phi_2)$.

Similarly, $L(\lambda\phi_1) = \lambda\phi_1 \frac{d}{dx} (\lambda\phi_1) = \lambda^2\phi_1 \frac{d\phi_1}{dx} = \lambda^2 L(\phi_1) \neq \lambda L(\phi_1)$. Thus $L(\lambda\phi_1) \neq \lambda L(\phi_1)$ and L is nonlinear. It is noted that if either of the two linear operator relationships fail, then the operator L is said to be nonlinear.

4.4. Linear Operator Norms

In our typical approximation problem setting, there exists a linear space $X = S^m$ which is composed of elements $\phi \in S^m$ such that $\phi = \sum_{j=1}^{m} \lambda_j f_j$ where $\lambda_j \in \mathbf{R}$ and $\{f_j\}$ is the set of m selected basis functions; that is $\text{Dim}(S^m) = m$. The problem to be approximated is a linear operator equation $L(\phi) = h$ where L is a linear operator.

Generally, any element $\phi \in S^m$ does not satisfy the linear operator equation and so there exists approximation error in the input vector. Of concern is how this approximation error is magnified (or diminished) by L.

If ϕ^* and h^* are the true solutions to $L(\phi) = h$, then the approximation error is $e = \phi^* - \phi$ and $L(e) = L(\phi^* - \phi) = L(\phi^*) - L(\phi) = h^* - h$. To quantify the magnification of approximation due to the linear operator, the norm of the operator is defined by

$$||L|| = \sup\{||Le||/||e||: e \in S \text{ and } e \neq 0\} \tag{4.1}$$

Because $||\lambda e|| = \lambda||e||$, $L(\lambda e) = \lambda L(e)$ and $||L(\lambda e)|| = \lambda||L(e)||$, (4.1) can be rewritten as

$$||L|| = \sup\{||L(\lambda e)||/||\lambda e||: e \in S \text{ and } e \neq 0\} \tag{4.2}$$

Choosing $\lambda = 1/||e||$ gives

$$||L|| = \sup\{||Le||: \ ||e|| = 1\} \tag{4.3}$$

But the error vector is transformed by L exactly as any vector $s \in S$ (e is an element on S), therefore

$$||L|| = \sup\{||L\phi||: \ ||\phi|| = 1\} \tag{4.4}$$

It is noted that the norm of the output space T is used to evaluate $||L(\phi)||$ whereas the norm of the input space S is used to evaluate $||\phi||$.

EXAMPLE 4.9.

The mathematical relationship being considered is the integral equation $f(x) = \int_0^x \lambda\phi(t)dt$ where $x \in [0,1]$, $\phi(t) \in C[0,1]$, and $\lambda \in \mathbf{R}$ such that $\lambda > 0$. Then $L(\phi) = \int_0^x \lambda\phi(t)dt$. Using the maximum norm for $C[0,1]$, $||\phi(t)|| = \max\{|\phi(t)|: 0 \leq t \leq 1\}$. Then $||L(\phi)||$ will be a maximum when $x = 1$ and $||L(\phi)|| = (1)\lambda||\phi(t)||$. Thus, $||L|| = \lambda$.

CHAPTER 5

THE BEST APPROXIMATION METHOD

5.0. Introduction

Many important engineering problems fall into the category of being linear operators, with supporting boundary conditions. In this chapter, an inner-product and norm is developed which enables the engineer to approximate such engineering problems by developing a generalized Fourier series. The resulting approximation is the "best" approximation in that a least-squares (L_2) error is minimized simultaneously for fitting both the problem's boundary conditions and satisfying the linear operator relationship (the governing equations) over the problem's domain (both space and time). Because the numerical technique involves a well-defined inner product, error evaluation is readily available using Bessel's inequality. Minimization of the approximation error is subsequently achieved with respect to a weighting of the inner product components, and the addition of basis functions used in the approximation.

5.1. An Inner Product for the Solution of Linear Operator Equations

The general setting for solving a linear operator equation with boundary values by means of an inner product is as follows: Let Ω be a region in R^m with boundary Γ and denote the closure of Ω by $cl(\Omega)$. Consider the real Hilbert space $L_2(cl(\Omega), d\mu)$, which has

inner product $(f,g) = \int fg d\mu$. To construct the inner product for the development of a generalized Fourier Series is to choose the measure μ correctly; that is let μ be one measure μ_1 on Ω and another measure μ_2 on Γ. One choice for a plane region would be for μ_1 to be the usual two dimensional Lebesque measure $d\Omega$ on Ω and for μ_2 to be the usual arc length measure $d\Gamma$ on Γ. Then an inner product is given by

$$(f,g) = \int_{\Omega} fg \, d\Omega + \int_{\Gamma} fg \, d\Gamma \qquad (5.1)$$

Consider a boundary value problem consisting of an operator L defined on domain D(L) contained in $L_2(\Omega)$ and mapping into $L_2(\Omega)$, and a boundary condition operator B defined on a domain D(B) in $L_2(\Omega)$ and mapping it into $L_2(\Gamma)$. The domains of L and B have to be choosen so at least for f in D(L), Lf is in $L_2(\Omega)$, and for f in D(B), Bf is in $L_2(\Gamma)$. For example we could have $Lf = \nabla^2 f$, and Bf(s) equal the almost everywhere (a.e.) radial limit of f at the point s on Γ, with appropriate domains.

The next step is to construct an operator T mapping its domain $D(T) = D(L) \cap D(B)$ into $L_2(cl(\Omega))$ by (for example, Davis and Rabinowitz, 1961)

$$\begin{aligned} Tf(x) &= Lf(x) \text{ for } x \text{ in } \Omega \\ Tf(s) &= Bf(s) \text{ for } s \text{ on } \Gamma. \end{aligned} \qquad (5.2)$$

From (5.2), there exists a single operator T on the Hilbert space $L_2(cl(\Omega))$ which incorporates both the operator L and the boundary conditions B, and which is linear if both L and B are linear.

Consider the inhomogeneous equation $Lf = g_1$ with the inhomogeneous boundary conditions $Bf = g_2$. Then define a function g on $cl(\Omega)$ by $g = g_1$ on Ω and $g = g_2$ on Γ. Then if the solution exists for the operator equation $Tf = g$, the solution f satisfies $\nabla^2 f = g_1$ on Ω, $f = g_2$ on Γ in the usual sense of meaning that the radial limit of f is g_2 on Γ. One way to attempt to solve the equation $Tf = g$ is to look at a subspace D_n of dimension n, which is contained in D(T), and to try to minimize $||Th - g||$ over all the h in D_n.

In this chapter, the mathematical development of the Best Approximation Method is presented. Three simple but detailed example problems are included to illustrate the inner products employed in the method, and to demonstrate the progression of steps used in the development of the associated computer program. Extension of the Best Approximation Method to a computer program for the approximation of boundary value problems of the two-dimensional Laplace equation is contained in Chapter 6. Generalization of the computer program to other linear operator problems is the focus of the final chapter.

5.1.1. Definition of Inner Product and Norm

Given a linear operator relationship

$$L(\phi) = h \text{ on } \Omega, \ \phi = \phi_b \text{ on } \Gamma \tag{5.3}$$

defined on the problem domain Ω with auxilliary conditions of $\phi = \phi_b$ on the boundary Γ(see Fig. 5.1). Here Ω may represent both time and space, and ϕ_b may be both initial and boundary conditions. It is assumed that the working space is sufficiently restricted (see following) such that ϕ is a unique almost everywhere (ae) solution to (5.3).

Choose a set of m linearly independent functions $<f_j>^m$, and let S^m be the m-dimensional space spanned by the elements of $<f_j>^m$. Here, the elements of $<f_j>^m$ will be assumed to be functions of the dependent variables appearing in (5.3).

An inner-product is defined for elements of S^m by (u,v) where for $u,v \in S^m$

$$(u,v) = \int_\Gamma uv d\Gamma + \int_\Omega LuLv d\Omega \qquad (5.4)$$

It is seen that (u,v) is indeed an inner-product, because for elements u,v,w in S^m

(i) $(u,v) = (v,u)$

(ii) $(ku,v) = k(u,v)$, for L a linear operator

(iii) $(u+v,w) = (u,w) + (v,w)$ for L a linear operator

(iv) $(u,u) = \int_\Gamma (u)^2 d\Gamma + \int_\Omega (Lu)^2 d\Omega \geq 0$

(v) $(u,u) = 0 \Longrightarrow u = 0$ ae on Γ, and $Lu = 0$ ae over Ω

The above restrictions on the operator L imply that L is linear (see (ii) and (iii) in the above definition); if $Lu = 0$ ae over Ω and $u = 0$ ae on Γ, this must imply that the solution $u = [0]$, where [0] is the zero element over $\Omega \cup \Gamma$; and for the inner-product to exist, the integrals must exist. For the inner-product of (5.4) to exist, the integrands must be finite. Additionally, each element $u \in S^m$ must satisfy $\int_\Gamma u^2 d\Gamma < \infty$.

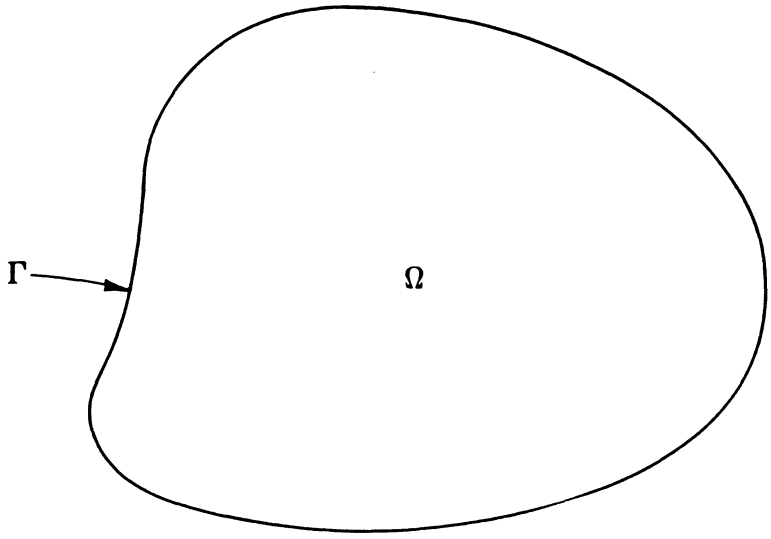


Fig. 5.1: Definition of Problem Domain, Ω, and Boundary, Γ. (NOTE: ϕ_b can include the temporal term boundary of the initial condition specification.)

For the above restrictions of L and the space S^m, the inner-product is defined and a norm "|| ||" immediately follows,

$$||u|| \equiv (u,u)^{\frac{1}{2}} \tag{5.5}$$

The generalized Fourier series approach can now be used to obtain the "best" approximation $\phi_m \in S^m$ of the function ϕ using the newly defined inner-product and corresponding norm presented in (5.4) and (5.5).

The next step in developing the generalized Fourier series is to construct a new set of functions $<g_j>^m$ which are the orthonormal representation of the $<f_j>^m$.

5.2. Orthonormalization Process

The functions $<g_j>^m$ can be obtained by the well-known Gramm-Schmidt procedure (Kantorovich and Krylor, pg. 45, 1964) using the newly defined norm of (5.4). That is,

$$\begin{aligned} g_1 &= f_1/||f_1|| \\ &\vdots \\ g_m &= [f_m - (f_m,g_1)g_1 - \cdots - (f_m,g_{m-1})g_{m-1}]/ \\ &\qquad ||f_m - (f_m,g_1)g_1 - \cdots - (f_m,g_{m-1})g_{m-1}|| \end{aligned} \tag{5.6}$$

The modified Gramm-Schmidt process first modifies all the $f_j (j \geq 2)$ to be orthogonal with respect to f_1, and then normalizes f_1 to obtain g_1. The next step is to make the modified f_j' $(j \geq 3)$ orthogonal to f_2', and then normalize f_2' to obtain g_2. Continuing in this fashion, the orthonormal set $\{g_j\}$ is determined.

Hence, the elements of $\langle g_j \rangle^m$ satisfy the convenient properties that

$$(g_j, g_k) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} \tag{5.7}$$

In a subsequent section, a simple one-dimensional problem illustrates the orthonormalization procedure of (5.6).

The elements $\langle g_j \rangle^m$ also form a basis for S^m but, because of (5.7), can be directly used in the development of a generalized Fourier series where the computed coefficients do not change as the dimension m of $\langle g_j \rangle^m$ increases. That is, as the number of orthonormalized elements increases in the approximation effort, the previously computed coefficients do not change. Each element $\phi_m \varepsilon S^m$ can now be written as

$$\phi_m = \sum_{j=1}^{m} \gamma_j g_j \ , \quad \phi_m \varepsilon S^m \tag{5.8}$$

where γ_j are unique real constants.

5.3. Generalized Fourier Series

The ultimate objective is to find the element $\phi_m \varepsilon S^m$ such that $||\phi_m - \phi||$ is a minimum. That is, we want $||\phi_m - \phi||^2$ to be a minimum, where

$$||\phi_m - \phi||^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(L \sum_{j=1}^{m} \gamma_j g_j - L\phi \right)^2 d\Omega \tag{5.9}$$

Remembering that L is a linear operator, and $L\phi = h$ by the problem definition of (5.3), then (5.9) can be rewritten as

$$||\phi_m - \phi||^2 = \int_\Gamma \left(\sum_{j=1}^{m} \gamma_j g_j - \phi_b \right)^2 d\Gamma + \int_\Omega \left(\sum_{j=1}^{m} \gamma_j Lg_j - h \right)^2 d\Omega \tag{5.10}$$

Thus, minimizing $||\phi_m - \phi||^2$ is equivalent to minimizing the error of approximating the boundary conditions and the error of approximating the governing operator relationship in a least-square (or L_2) sense. Because the $<g_j>^m$ are orthonormalized and the inner-product (,) is well-defined, the coefficients γ_j of (5.8) are immediately determined by the generalized Fourier constants, γ_j^*, where

$$\gamma_j^* = (g_j, \phi) \ , \quad j = 1,2,\cdots,m \tag{5.11}$$

Thus

$$\phi_m^* = \sum_{j=1}^{m} \gamma_j^* g_j = \sum_{j=1}^{m} (g_j, \phi) g_j \tag{5.12}$$

is the "best" approximation of ϕ, in the space S^m.

EXAMPLE 5.1: (Differential Equation)

To illustrate the previous developments, a simple one-dimensional torsion problem is studied. In this example, four polynomials (linearly independent functions) are used as a basis,

$$<f_j>^4 = <1, x, x^2, x^3>$$

with the problem defined by, for $0 \leq x \leq 1$,

$$\frac{d^2\phi}{dx^2} = -2 \ , \quad \phi(x=0) = 1 \text{ and } \phi(x=1) = 2.$$

Here $L = \frac{d^2}{dx^2}$, $h = -2$, and ϕ_b is given by the two point values at $x = 0$ and 1. The inner-product of (5.4) is now given as

$$(u,v) = \int_\Gamma uv d\Gamma + \int_\Omega LuLv d\Omega = uv\Big|_{x=0} + uv\Big|_{x=1} + \int_\Omega \frac{d^2(u)}{dx^2} \frac{d^2(v)}{dx^2} d\Omega$$

The 4-dimension space S^4 is the set of all functions (polynomials) such that $\phi_4(x) = C_1 + C_2\, x + C_3\, x^2 + C_4\, x^3$ where the C_i are real constants.

The orthonormalization of the $\langle f_j \rangle^4$ proceeds as follows:

For element g_1:

$$(f_1,f_1) = (1)(1)\Big|_{x=0} + (1)(1)\Big|_{x=1} + \int_{x=0}^{1} \frac{d^2(1)}{dx^2}\,\frac{d^2(1)}{dx^2}\,dx = 2$$

and

$$g_1 = f_1 \,/\, ||f_1|| = 1/\sqrt{2} = \sqrt{2}/2$$

For element g_2:

$$(f_2,g_1) = (x,\sqrt{2}/2) = (x)(\sqrt{2}/2)\Big|_{x=0} + (x)(\sqrt{2}/2)\Big|_{x=1} + \int_{0}^{1} \frac{d^2(x)}{dx^2}\,\frac{d^2(\sqrt{2}/2)}{dx^2}\,dx$$

$$= \sqrt{2}/2$$

$$\hat{g}_2 = f_2 - (f_2,g_1)g_1 = x - (\sqrt{2}/2)(\sqrt{2}/2) = x - 1/2$$

$$(\hat{g}_2,\hat{g}_2) = (x-1/2)(x-1/2)\Big|_{x=0} + (x-1/2)(x-1/2)\Big|_{x=1} + \int_{0}^{1} \frac{d^2(x-1/2)}{dx^2}\,\frac{d^2(x-1/2)}{dx^2}\,dx$$

$$= 1/2$$

$$\therefore g_2 = \hat{g}_2 \,/\, ||\hat{g}_2|| = (x-1/2) \,/\, (\sqrt{2}/2) = (2x-1)/\sqrt{2}$$

Similarly for element g_3:

$$g_3 = (x^2 - x)/2$$

Element g_4 is more involved and is derived in detail for illustration:

$$(f_4,g_1) = \sqrt{2}/2 \; ; \quad (f_4,g_2) = \sqrt{2}/2 \; ; \quad (f_4,g_3) = 3$$

$$\hat{g}_4 = x^3-(f_4,g_1)g_1 - (f_4,g_2)g_2 - (f_4,g_3)g_3 = x^3 - \frac{3}{2}x^2 + \frac{x}{2}$$

$$(\hat{g}_4,\hat{g}_4) = \left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right)^2\Big|_{x=0} + \left(x^3 - \frac{3}{2}x^2 + \frac{x}{2}\right)^2_{x=1} + \int_0^1 \left[\frac{d^2(x^3 - \frac{3}{2}x^2 + \frac{x}{2})}{dx^2}\right]^2 dx$$

$$= 0 + 0 + \int_0^1 (6x - 3)^2 \, dx = 3$$

$$\therefore g_4 = \hat{g}_4 \Big/ ||g_4|| = (x^3 - \frac{3}{2}x^2 + \frac{1}{2}x)\Big/\sqrt{3}$$

Hence, the orthonormal vectors $\langle g_j \rangle^4$ are

$$\langle g_j \rangle^4 = \langle \sqrt{2}/2, \; (2x-1)/\sqrt{2}, \; (x^2 - x)/2, \; (2x^3 - 3x^2 + x)/2\sqrt{3} \rangle$$

Now, any element $\phi_4 \in S^4$ is of the form

$$\phi^4 = \sum_{j=1}^{4} \gamma_j g_j$$

The norm $||\phi_4 - \phi||$ is a minimum when $\gamma_j = \gamma_j^*$ where γ_j^* are the generalized Fourier series coefficients determined from

$$\gamma_j^* = (g_j,\phi).$$

That is,

$$\gamma_j^* = \int_\Gamma g_j \phi_b d\Gamma + \int_\Omega Lg_j \, L\phi d\Omega = \int_\Gamma g_j \phi_b d\Gamma + \int_\Omega Lg_j h d\Omega$$

where for simplicity, the Lg_j are given by

$$\langle Lg_j \rangle^4 = \langle 0, \; 0, \; 1, \; (6x-3)/\sqrt{3} \rangle$$

Remembering that h = -2 by the problem definition, we solve for the γ_j^* as follows:

$$\gamma_1^* = (g_1,\phi) = \left(\frac{\sqrt{2}}{2}\right)\left(\phi_b\right)\Big|_{x=0} + \left(\frac{\sqrt{2}}{2}\right)\left(\phi_b\right)\Big|_{x=1} + 0 = 3\sqrt{2}/2$$

$$\gamma_2^* = (g_2,\phi) = \left(\frac{2x-1}{\sqrt{2}}\right)\left(\phi_b\right)\Big|_{x=0} + \left(\frac{2x-1}{\sqrt{2}}\right)\left(\phi_b\right)\Big|_{x=1} + 0 = \sqrt{2}/2$$

$$\gamma_3^* = (g_3,\phi) = 0 + 0 + \int_0^1 Lg_3 f dx = \int_0^1 (1)(-2)dx = -2$$

$$\gamma_4^* = (g_4,\phi) = 0 + 0 + \int_0^1 Lg_4 f dx = \int_0^1 \left(\frac{6x-3}{\sqrt{3}}\right)(-2)\,dx = 0$$

Thus, the best approximation in S^4 is given by

$$\phi_4^* = \sum_{j=1}^{4} \gamma_j^* g_j = 1 + 2x - x^2$$

It is readily seen that $L\phi_4^* = -2 = h$, and ϕ_4^* satisfies the problem boundary conditions.

5.3.1. Discussion

From the example problem, a best approximation of a linear operator relationship is obtained by a generalized Fourier series development which minimizes, in a least-squares (L_2) sense, the error of approximation.

Because the generalized Fourier series approach is used, several advantages over a matrix solution (for the generalized Fourier series coefficients) are obtained:

1. Elimination of the need for solving large, fully populated, matrices such as occurs when solving the normal equations.
2. Elimination of the instability which typically arises in a matrix solution for Fourier coefficients (i.e., higher powers of the expansion basis functions assumed).

3. The generalized Fourier series coefficients do not change as additional functions are added (i.e., as the dimension m of the space S^m is increased).
4. Generalized Fourier series theory applies; hence, error analysis can be conducted using Bessel's inequality as discussed in the next section.

5.4. Approximation Error Evaluation

Due to the generalized Fourier series approach and the definition of the inner-product, Bessel's inequality applies. That is, for any dimension m

$$(\phi,\phi) \geq \sum_{j=1}^{m} (g_j,\phi)^2 = \sum_{j=1}^{m} \gamma_j^{*2} \tag{5.14}$$

where

$$(\phi,\phi) = \int_{\Gamma} (\phi)^2 d\Gamma + \int_{\Omega} (L\phi)^2 d\Omega = \int_{\Gamma} \phi^2 d\Gamma + \int_{\Omega} h^2 d\Omega \tag{5.15}$$

Equation (5.15) is readily evaluated and forms an upper bound to the sum of $(g_j,\phi)^2$ as the dimension m increases. Consequently, one may interact with the approximation effort by carefully adding functions to the $\langle f_j \rangle^m$ in order to best reduce the difference computed by Bessel's inequality. In a following section, Bessel's inequality will be used to define an objective function (noted by χ) which will be subsequently minimized by determining a weighting factor ε to be used in the inner product of (5.4).

EXAMPLE 5.2: (Voltera Integral)

To further illustrate the approximation method, a Voltera integral equation (such as occurs in developing unit hydrographs from watershed rainfall-runoff data) is considered where

$$q(t) = \int_0^t i(t-s)\phi(s)ds, \quad 0 \le t \le 2$$

where for simplicity the effective rainfall intensity is given by the constant value

$$i(t-s) = 1$$

and the runoff hydrograph flowrate q(t) is given by

$$q(t) = \begin{cases} t^3, & 0 \le t \le 1 \\ -2t^2 + 7t - 4, & 1 \le t \le 2 \end{cases}$$

In this class of problem, neither boundary (nor initial) conditions are involved, hence the inner product of (5.4) becomes

$$(u,v) = \int_\Omega LuLv d\Omega$$

$$= \int_{t=0}^{2} \left[\int_0^t i(t-s)u(s)ds \int_0^t i(t-s)v(s)ds \right] dt \qquad (5.16)$$

By assumption i(t-s) = 1, and the inner product reduces to

$$(u,v) = \int_{t=0}^{2} \left[\int_0^t u(s)ds \int_0^t v(s)ds \right] dt$$

Three elements are considered for basis functions $\langle f_j \rangle^3$, namely the polynomials $\langle 1, s, s^2 \rangle$. The orthonormalized elements $\langle g_j \rangle^3$ are determined in the following:

g_1:

$$Lf_1 = \int_0^t (1)ds = t$$

$$\therefore (f_1, f_1) = \int_0^2 t^2 dt = 8/3; \quad ||f_1|| = 2\sqrt{\frac{2}{3}}$$

$$\text{and } g_1 = f_1/||f_1|| = \sqrt{\frac{3}{8}}$$

g_2:

$$Lf_2 = \int_0^t sds = t^2/2$$

$$Lg_1 = \int_0^t \sqrt{\frac{3}{8}}\, ds = t\sqrt{\frac{3}{8}}$$

$$\therefore (f_2, g_1) = \int_0^2 Lf_2 Lg_1 dt = \int_0^2 \left(\frac{t^2}{2}\right)\left(t\sqrt{\frac{3}{8}}\right) dt = \sqrt{\frac{3}{2}}$$

Now $\hat{g}_2 = f_2 - (f_2, g_1)g_1 = s - 3/4$

$$L\hat{g}_2 = \int_0^t (s - 3/4)\, ds = \frac{t^2}{2} - \frac{3t}{4}$$

$$\therefore (\hat{g}_2, \hat{g}_2) = \int_0^2 \left(\frac{t^2}{2} - \frac{3t}{4}\right)^2 dt = \frac{1}{10}$$

$$\therefore g_2 = \hat{g}_2/||\hat{g}_2|| = \left(s - \frac{3}{4}\right)\sqrt{10}$$

g_3:

Analogous to the above,

$$(f_3,g_1) = \frac{16}{5}\sqrt{\frac{1}{6}}$$

$$(f_3,g_2) = \sqrt{10}/5.625$$

$$\therefore \hat{g}_3 = f_3 - (f_3,g_1)g_1 - (f_3,g_2)g_2$$

$$= s^2 + 0.5\bar{3} - 1.\bar{7}s$$

where the overbar notation indicates repetitive digits. Finally,

$$g_3 = \hat{g}_3/||\hat{g}_3|| = 10.5234s^2 - 18.708s + 5.6125$$

The generalized Fourier coefficients are determined as before by

$$\gamma_1^* = (g_1,\phi) = \int_{\Omega} Lg_1 L\phi d\Omega$$

$$= \sqrt{\frac{3}{8}} \int_0^1 (t)(t^3)dt + \sqrt{\frac{3}{8}} \int_1^2 (t)(-2t^2 + 7t-4)dt = 1.8575$$

$$\gamma_2^* = (g_2,\phi) = 0.21082$$

$$\gamma_3^* = (g_3,\phi) = -0.325$$

Thus the best approximation is developed (for the defined inner product of (5.16)) by

$$\phi_3 = -3.42s^2 + 6.7467s - 1.1865$$

For this example problem, the exact solution is determined by taking the derivative of the q(t) function (rewritten in terms of the variable s)

$$\phi(s) = \begin{cases} 3s^2, & 0 \le s \le 1 \\ -4s+7, & 1 \le s \le 2 \end{cases}$$

Figure 5.2 compares the exact solution $\phi(s)$ to the approximation function $\phi_3(s)$ developed from using only 3 polynomial basis functions.

It is noted that although the EXAMPLE 5.1 and EXAMPLE 5.2 are different operator relationships (i.e., a differential equation and a Voltera integral), the approximation method and procedures are identical.

Additionally, Bessel's inequality can be used to evaluate the error of approximation for this problem as follows:

$$(\phi,\phi) = \int_{t=0}^{2} (L\phi)^2 dt = \int_{t=0}^{2} [q(t)]^2 dt = \int_{0}^{1} (t^3)^2 dt + \int_{1}^{2} (-2t^2 + 7t - 4)^2 dt = 3.6095$$

In comparison,

$$\sum_{j=1}^{m} \gamma_j^{*2} = (1.8575)^2 + (0.21082)^2 + (-0.325)^2 = 3.6003 \le (\phi,\phi)$$

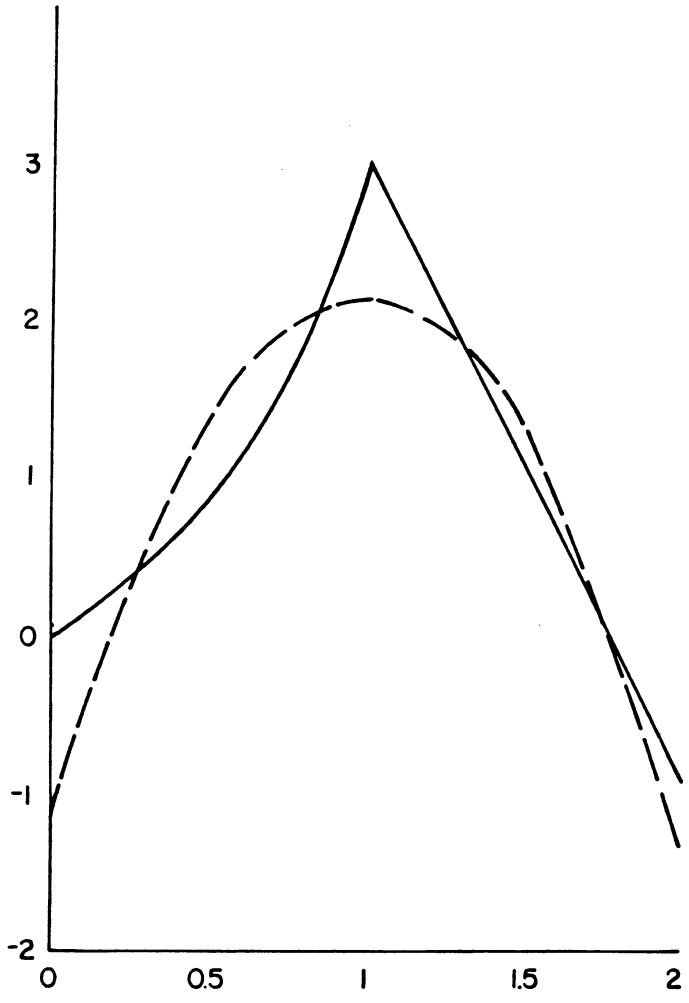


Fig. 5.2: Best Approximation Unit Hydrograph (dashed) and Exact Unit Hydrograph (solid line).

That is, although the generalized Fourier coefficients provide for the best approximation from space S^m, the error of approximation χ given by

$$\chi = (\phi,\phi) - \sum_{j=1}^{m} \gamma_j^{*2} \tag{5.17}$$

is nonzero and, therefore, the addition of additional elements to $\langle f_j \rangle^m$ (increasing the dimension of S^m) will necessarily add more positive values to the sum the γ_j^{*2}, resulting in a decrease in χ. Should $\chi = 0$, then $||\phi - \phi_m|| = 0$ and $\phi - \phi_m = [0]$, the zero element, and $\phi = \phi_m$ ae.
For instance, EXAMPLE 5.1 results in

$$(\phi,\phi) = \int_{\Gamma} (\phi)^2 d\Gamma + \int_{\Omega} (L\phi)^2 d\Omega$$

$$= (\phi_b)^2 \Big|_{x=0} + (\phi_b)^2 \Big|_{x=1} + \int_{x=0}^{1} (h)^2 dx$$

$$= (1) + (4) + (4) = 9$$

In comparison,

$$\sum_{j=1}^{m} \gamma_j^{*2} = \sum_{j=1}^{4} \gamma_j^{*2} = \left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (-2)^2 + (0)^2 = 9$$

Thus, $\chi = 0$, which indicates that the approximation ϕ_4 equals the exact solution ϕ ae (almost everywhere). Of course for this example, $\phi = \phi_4$ identically over Ω, and the ae statement can be dropped.

5.5. The Weighted Inner Product

In the inner product of (5.4), equal weight is given to the various requirements imposed on the best approximation function ϕ_m from the space S^m spanned by the m linearly independent basis functions $\langle f_j \rangle^m$. Namely, the L_2 error in satisfying the linear operator relationship over Ω is considered by equal importance as the L_2 error in satisfying the problem's boundary (and initial) conditions, (of course for the Voltera integral example problem, only one term is used in the inner product definition and the concerns as to weighting factors is no longer needed).

Due to the limitations of computer power, only a finite number of basis functions can be used for approximation purposes, and so an argument is made to weight the terms which compose the inner product differently. For $0<\varepsilon<1$, one weighting of (5.4) is simply

$$(u,v) = \varepsilon \int_{\Gamma} uv d\Gamma + (1-\varepsilon) \int_{\Omega} LuLv d\Omega \qquad (5.18)$$

In (5.18), an ε-value close to 1 would force the approximation function ϕ_n of S^m to focus upon satisfying the problem's boundary conditions rather than satisfying the operator in Ω. Similarly, and ε-value close to 0 would focus the ϕ_m approximation towards satisfying the operator relationship in Ω and ignore the boundary conditions.

It is noted that (5.18) is still an inner product for a given choice of ε, and will be used to develop the generalized Fourier series using the previously presented procedures. And as the dimension S^m increases, the Bessel's inequality still applies in that $\chi = \chi_\varepsilon$, and

$$\chi_\varepsilon = 0 \Rightarrow ||\phi_m - \phi||_\varepsilon = 0 \qquad (5.19)$$

In (5.19), ε-notation has been added to clarify that all norms, inner products, and even the orthonormalized basis functions are now functions of ε for $0<\varepsilon<1$. However for ease of presentation in the following text, the ε-notation is omitted although it is implied that all relationships are now dependent on the ε-value used in the weighting of the inner product components.

The selection of the "optimum" ε-value to be used in (5.18) depends on the rule assigned for optimization. In this paper, ε is chosen which minimizes the Bessel's inequality relationship

$$\chi_\varepsilon = (\phi,\phi)_\varepsilon - \sum_{j=1}^{m} \gamma^{*2}_{\varepsilon_j} \tag{5.20}$$

$$= (\phi,\phi)_\varepsilon - \sum_{j=1}^{m} (\phi, g_{\varepsilon_j})^2_\varepsilon \tag{5.21}$$

In (5.20) and (5.21) it is stressed that all terms depend on ε. The inner product weighting ε-value is chosen which minimizes χ_ε of (5.21).

EXAMPLE 5.3: (Weighted Inner Product)

To illustrate the inner product weighting concept, EXAMPLE 5.1 is restudied with only one basis function, $f_1 = x^2$. It is recalled that $L\phi = \frac{d^2\phi}{dx^2}$, $h = -2$, and $\phi(x=0) = 1$, $\phi(x=1) = 2$. Proceeding as before, and dropping the ε subscript notation,

$$(f_1,f_1) = \varepsilon \int_{\Gamma} (f_1)^2 d\Gamma + (1-\varepsilon) \int_{\Omega} (Lf_1)^2 d\Omega$$

$$= \varepsilon (x^2)^2 \Big|_{x=0} + \varepsilon (x^2)^2 \Big|_{x=1} + (1-\varepsilon) \int_0^1 (-2)^2 dx = 4 - 3\varepsilon$$

$$\therefore \; ||f_1|| = \sqrt{4-3\varepsilon}$$

and $g_1 = f_1 / ||f_1|| = x^2/\sqrt{4-3\varepsilon}$

The only Fourier coefficient γ_1^* is computed as

$$\gamma_1^* = (\phi,g_1) = \varepsilon \left\{ \frac{x^2}{\sqrt{4-3\varepsilon}} \right\}(1) \Big|_{x=0} + \varepsilon \left\{ \frac{x^2}{\sqrt{4-3\varepsilon}} \right\}(2) \Big|_{x=1}$$

$$+ (1-\varepsilon) \int_0^1 \left(\frac{2}{\sqrt{4-3\varepsilon}} \right)(-2)dx = \frac{(6\varepsilon-4)}{\sqrt{4-3\varepsilon}}$$

Thus, $\phi_1 = \gamma_1 g_1$

$$= x^2 \left(\frac{6\varepsilon-4}{4-3\varepsilon} \right) \; ; \quad \text{for } 0<\varepsilon<1$$

The next step is to compute χ_ε:

$$(\phi,\phi) = \varepsilon(\phi_b)^2 \Big|_{x=0} + \varepsilon(\phi_b)^2 \Big|_{x=1} + (1-\varepsilon) \int_0^1 h^2 dx$$

$$= \varepsilon(1)^2 + \varepsilon(2)^2 + (1-\varepsilon) \int_0^1 (-2)^2 dx$$

$$= 4+\varepsilon; \text{ for } 0<\varepsilon<1$$

$$\gamma_1^{*2} = (36\varepsilon^2 - 48\varepsilon + 16)/(4-3\varepsilon)$$

Therefore

$$\chi_\varepsilon = (\phi,\phi) - \gamma_1^{*2}$$

$$= (4+\varepsilon) - (36\varepsilon^2 - 48\varepsilon + 16)/(4-3\varepsilon)$$

$$= 4 + 13\varepsilon + 16/(3\varepsilon-4)$$

$$= \varepsilon(39\varepsilon - 40)/(3\varepsilon-4)$$

Figure 5.3 displays the plot of χ_ε against ε for $0<\varepsilon<1$. Because only one basis function $f_1 = x^2$ was chosen in this simple example, the weighting is focused on satisfying the operator or the boundary conditions as shown in Table 5.1. For this simple problem, $\phi_1 = kx^2$ where $k = (6\varepsilon-4)/(4-3\varepsilon)$ from the above calculations. Table 5.1 summarizes the implications resulting from using various values of k in ϕ_1.

Table 5.1. Inner Product Weighting Implications for EXAMPLE 5.3

ε	k(for $\phi = kx^2$)	Notes
0	-1.0	All weighting is focused toward satisfying $\frac{d^2\phi}{dx^2} = -2$. Here, $\phi_1 = -x^2$
0.50	-0.40	An intermediate value for ϕ_1
1.0	+2.0	All weighting is focused towards satisfying $\phi(x=0) = 1$ and $\phi(x=1) = 2$. Here $\phi_1 = 2x^2$.

From Fig. 5.3 it is seen that χ_ε is minimum when $\varepsilon = 0$. Obviously from Table 5.1, however, $\varepsilon = 0$ would not be the optimum choice of ε due to the approximation only satisfying in a minimum least-squares (L_2) sense the operator relationship and neglecting the boundary conditions. For typical applications, ε is chosen when maximizes χ_ε. In this way, the "largest" value of approximation error is being used to evaluate Bessel's inequality, which is then used to evaluate the reduction in approximation error as additional elements are added to the test collection of basis functions.

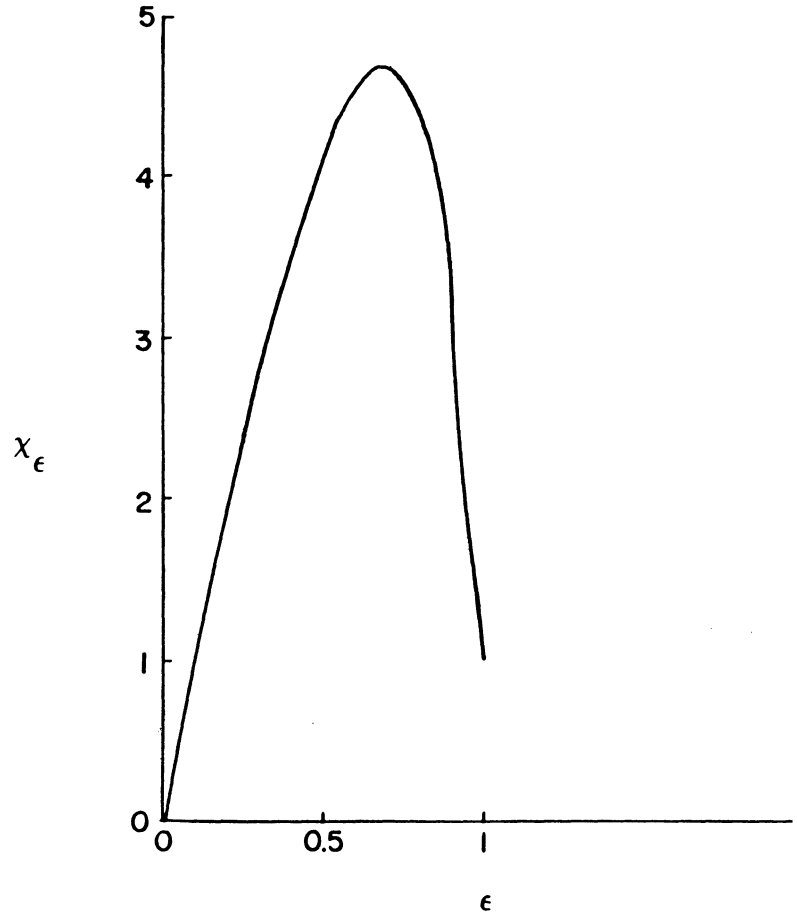


Fig. 5.3. $\chi_\varepsilon = (\phi,\phi) - \gamma_1^{*2}$ for Example Problem 5.3

CHAPTER 6

THE BEST APPROXIMATION METHOD: APPLICATIONS

6.0. Introduction

The theory of generalized Fourier Series as utilized in the Best Approximation Method can be applied to the approximation of linear operator relationships. To demonstrate the computational results in using this approach, several example problems where analytic solutions or quasi-analytic solutions exist are considered. Applications include two-dimensional problems involving the Laplace and Poisson equations, tests for variation in results due to inner product weighting factors, and applications to nonhomogeneous domain problems.

This chapter focuses on the topics of weighting factor selection and modeling sensitivity, effects of additional basis functions on computational accuracy, and the effects on modeling results due to the addition of collocation points. Thirteen simple but detailed example problems are used to illustrate the approximation results obtained by the method when applied to practical problems. A FORTRAN computer program based on the Best Approximation Method is included as a final section of this chapter.

6.1. Sensitivity of Computational Results to Variation in the Inner Product Weighting Factor

The inner product uses a weighting factor, ε, to weight the approximation effort in satisfying the PDE and BC values by

$$(u \cdot v) = \varepsilon \int_{\Gamma} u \cdot v \, d\Gamma + (1-\varepsilon) \int_{\Omega} Lu \, Lv \, d\Omega$$

The effects of varying ε between 0 and 1 is shown in the following simple application.

EXAMPLE 6.1: (Weighting Factor Sensitivity)

Let $\phi = 2x^2y + y^2 + x + 6$ where $\nabla^2\phi = 2 + 4y$ on the unit square domain. Figure 6.1 depicts the problem domain and boundary conditions.

First, define $<1,x,y,xy,x^2>$ to be the set of basis functions, and use $\varepsilon = 0$. The resulting approximation function is $\hat{\phi} = 2x^2$.

Applying the linear operator ∇^2 on $\hat{\phi}$, we obtain $\nabla^2\hat{\phi} = 4$. The graph of $\nabla^2\phi = 2 + 4y$ on the unit square domain is depicted in Fig. 6.2.

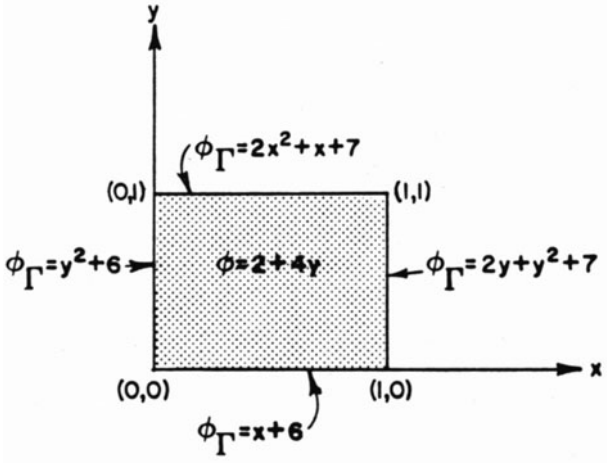


Fig. 6.1. Domain and Boundary Conditions for Example 6.1

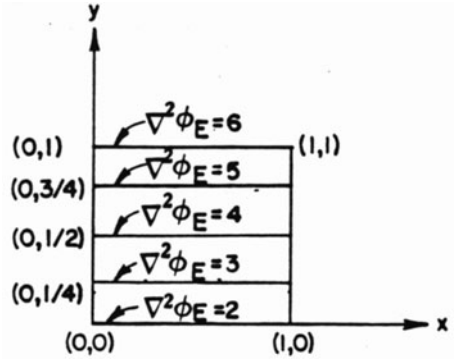


Fig. 6.2. Graphs for Exact Solution of $\nabla^2\phi = 2 + 4y$ on Unit Square Domain

From Fig. 6.2, it is seen that indeed for $\varepsilon = 0$, the best approximation for $\nabla^2\phi$ is $\nabla^2\phi = 4$ which coincides to $\nabla^2\hat{\phi} = 4$. Thusly, the $\nabla^2\hat{\phi}$ approximation satisfies the linear operator on the least square sense (L_2) for $\varepsilon = 0$ for the given set of basis functions.

Second, we choose <1,x,y> as the set of basis functions and use $\varepsilon = 1$. Then the resulting approximation function is $\hat{\phi} = 5.4583 + 2x + 1.75\,y$. For comparison, the least squares method can be used to minimize the function

$$\chi = \int_{\Gamma} (k_1 + k_2 x + k_3 y - \phi_\varepsilon)^2 d\Gamma$$

with respect to parameters k_1, k_2, and k_3 along the boundary of the unit square domain. From Fig. 6.1, one can write χ as follows:

$$\chi = \int_{x=0}^{1} (k_1 + k_2 x - x - 6)^2 dx + \int_{y=0}^{1} (k_1 + k_2 + k_3 y - 2y - y^2 - 7)^2 dy$$

$$+ \int_{x=0}^{1} (k_1 + k_2 x + k_3 - 2x^2 - x - 7)^2 dx + \int_{y=0}^{1} (k_1 + k_3 y - y^2 - 6)^2 dy$$

Minimizing χ with respect to k_1, k_2, and k_3, we obtain

$$k_1 = 5.4583$$

$$k_2 = 2.0$$

$$k_3 = 1.75$$

which verifies that the approximation function is the least square fit (L_2) with respect to the problem boundary conditions.

EXAMPLE 6.2: (Weighting Factor Sensitivity)

Let's consider a Poisson problem $\nabla^2\phi = 2 + 2y + 12x^2$ on an unit square domain with boundary condition $\phi_b = x^2 + y^2$. (Fig. 6.3)

By choosing the following set of basis functions $<x^2, x^3, x^4, y^2, y^3, y^4, xy^2, yx^2, x^2y^2>$, we obtain the approximation function $\hat{\phi}_b = x^2 + y^2$ for $\varepsilon = 1$ and $\hat{\phi}_\Omega = x^2 + \frac{1}{3}y^3 + x^4$ for $\varepsilon=0$ with $\nabla^2\hat{\phi}_\Omega = 2 + 2y + 12x^2$. This verifies the two extreme values of the weighting factor $\varepsilon=0$ and 1 in satisfying the PDE on the domain and satisfying the BC values on the boundary, respectively.

6.2. Solving Two-Dimensional Potential Problems

In the following are application problems and computed results in solving the Laplace equation in two-dimension. Because the Laplace equation is a linear operator, the Best Approximation Method approach is appropriate.

EXAMPLE 6.3. (Ideal Fluid Flow Around a 90° Bend)

The flow of an idea fluid around a 90° bend can be expressed by the analytic function

$$\omega = z^2$$
$$= (x^2 - y^2) + 2xyi$$

Since the state variable function $\phi = (x^2 - y^2)$ and the stream function $\psi=(2xy)$ are of polynomial forms, the approximation functions for the state variable and stream functions are found to result in the exact solutions regardless of the value of the weighting factor ε, $(0<\varepsilon<1)$. Figure 6.4 depicts the problem domain and nodal point placement used for this application.

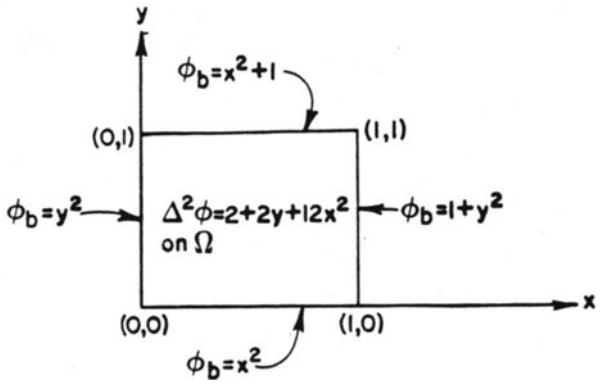


Fig. 6.3. Domain and Boundary Conditions for Example 6.2

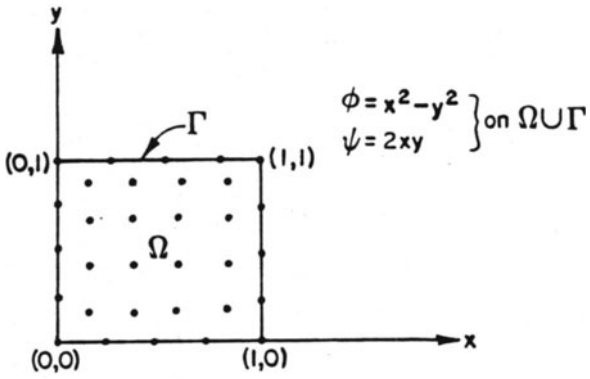


Fig. 6.4. Domain and Nodal Placement for Example 6.3

EXAMPLE 6.4: (St. Venant Torsion)

Consider the St. Venant torsion problem for an equilateral triangular section of Fig. 6.5. The analytic solution for $\phi(x,y)$ is given by

$$\phi(x,y) = (x^3 - 3xy^2)/2a + 2a^2/27$$

Figure 6.5 depicts the nodal placement and boundary conditions for the case of a = 3. The L_2 approximation function is found to result in the exact solution.

EXAMPLE 6.5: (Potential Problem: Sharp Corner Domain)

Figure 6.6 depicts the nodal placement and boundary conditions for a mechanical gear problem. The resulting L_2 approximation function is $\hat{\phi} = 9.518 + .3131x + .1812y + .7686xy + .2219x^2 - .2228y^2 - 1.018x^2y + .3397y^3 - .2463x^3y + .2124x^2y^2 + .2419xy^3$ for a weighting factor of $\epsilon = 0.5$. Table 6.1 compares the results from a Complex Variable Boundary Element Method or CVBEM (Hromadka, 1984) model and the L_2 approximation function for the interior nodal points, and the defined operator relationship.

EXAMPLE 6.6: (Ideal Fluid Flow Around a Cylinder)

Ideal fluid flow around a cylinder has the analytic function definition of

$$\omega(z) = z + \frac{1}{z}$$

The state variable (potential) function can be expressed as

$$\phi(x,y) = x \left[1 + \frac{1}{x^2 + y^2} \right]$$

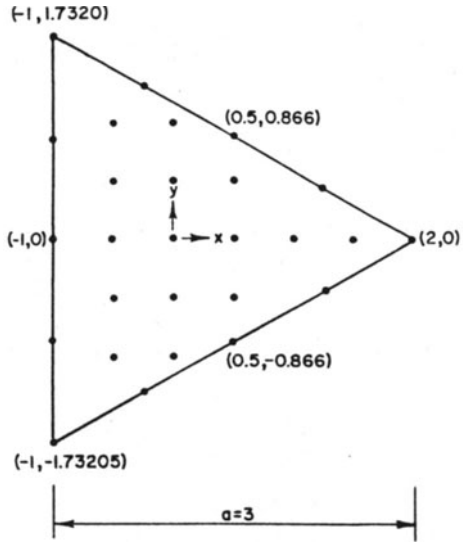


Fig. 6.5. Domain and Nodal Placement for Example 6.4

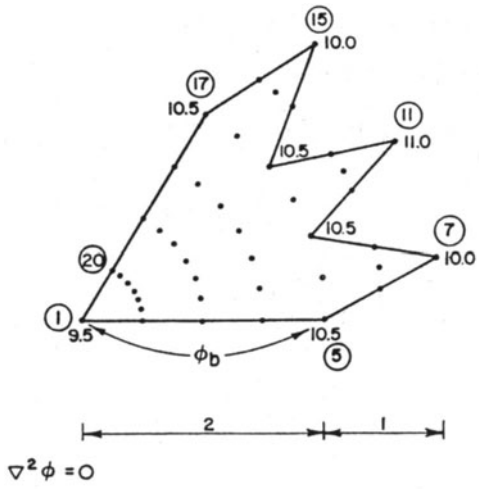


Fig. 6.6. Domain and Boundary Conditions for Example 6.5

Table 6.1. Comparison of L_2 and CVBEM Computational Results for EXAMPLE 6.5

Interior Nodal Points:

x	y	CVBEM	$\hat{\phi}$	$\nabla^2\hat{\phi}$
.321	.383	9.7842	9.752	8.03×10^{-5}
.643	.766	10.0391	10.04	8.56×10^{-4}
.964	1.149	10.2610	10.28	6.36×10^{-4}
1.286	1.532	10.3668	10.39	-5.79×10^{-4}
1.607	1.915	10.2417	10.31	-2.79×10^{-3}
.492	.087	9.857	9.752	2.35×10^{-4}
.985	.174	10.0992	10.04	1.19×10^{-3}
1.477	.261	10.2749	10.28	1.18×10^{-3}
1.970	.347	10.3659	10.39	-2.98×10^{-3}
2.462	.434	10.2439	10.31	-1.76×10^{-3}
.470	.171	9.8476	9.767	-3.55×10^{-4}
.940	.342	10.0944	10.05	-1.32×10^{-3}
1.410	.513	10.2830	10.27	-4.50×10^{-3}
.433	.250	9.8329	9.772	-5.88×10^{-4}
.866	.50	10.0839	10.05	-2.26×10^{-3}
1.299	.75	10.2856	10.27	-6.7×10^{-3}
1.732	1.0	10.4917	10.42	-1.39×10^{-2}
2.165	1.25	10.7489	10.63	-2.39×10^{-2}
.383	.321	9.8119	9.767	-4.34×10^{-4}
.766	.643	10.0659	10.05	-1.49×10^{-3}
1.149	.964	10.2806	10.27	-4.87×10^{-3}

and the stream function can be expressed as $\psi(x,y) = y\left\{1 - \frac{1}{x^2+y^2}\right\}$

where $\omega(z) = \phi(x,y) + i\psi(x,y)$. The L_2 approximation functions ($\varepsilon = 0.5$) are

$$\hat{\phi} = 1.704 + .9795x - 2.587y + 2.115xy - 1.009x^2 + 1.003y^2$$
$$- 1.289xy^2 + .431x^3 + .1452x^3y + .2258x^2y^2 - .1461xy^3$$

and

$$\hat{\psi} = -1.704 + 2.587x + 1.02y - 2.115xy - 1.003x^2 + 1.009y^2$$
$$+ 1.289x^2y - .4309y^3 + .1461x^3y - .2257x^2y^2 - 1.452xy^3$$

for state variable and stream functions, respectively. Figure 6.7 shows the approximation relative error between a CVBEM model and the L_2 approximation function values.

EXAMPLE 6.7: (Ideal Fluid Flow Around a Cylinder in 90° Bend)

Ideal fluid flow around a cylinder in a 90° bend has the analytic solution of $\omega(z) = z^2 + z^{-2}$. The state variable function ϕ and stream function ψ can be expressed, respectively, as

$$\phi = (x^2 - y^2)\left[\, 1 + \frac{1}{(x^2 + y^2)^2} \,\right]$$

and

$$\psi = 2xy \left[\, 1 - \frac{1}{(x^2 + y^2)^2} \,\right]$$

The L_2 approximation functions ($\varepsilon = 0.5$) for the state variable function $\hat{\phi}$ and the stream function $\hat{\psi}$ are given

$$\hat{\phi} = 3.197x - 3.197y - 1.828x^2 + 1.828y^2 + 1.894x^2y - 1.894xy^2$$
$$+ .6294x^3 - .6294y^3 + .4612x^3y - .4612xy^3$$

and

$$\hat{\psi} = -3.192 + 3.389x + 3.389y - 4.311xy + 1.352x^2 + 1.352y^2 + 2.121x^2y$$
$$+ 2.121xy^2 - .7136x^3 - .7136y^3 - .7536x^2y^2 + .1259x^4 + .1259y^4$$

on the domain shown in Fig. 6.8. In comparison of the approximation to exact values it was found that the relative error is high along the circumference of the cylinder.

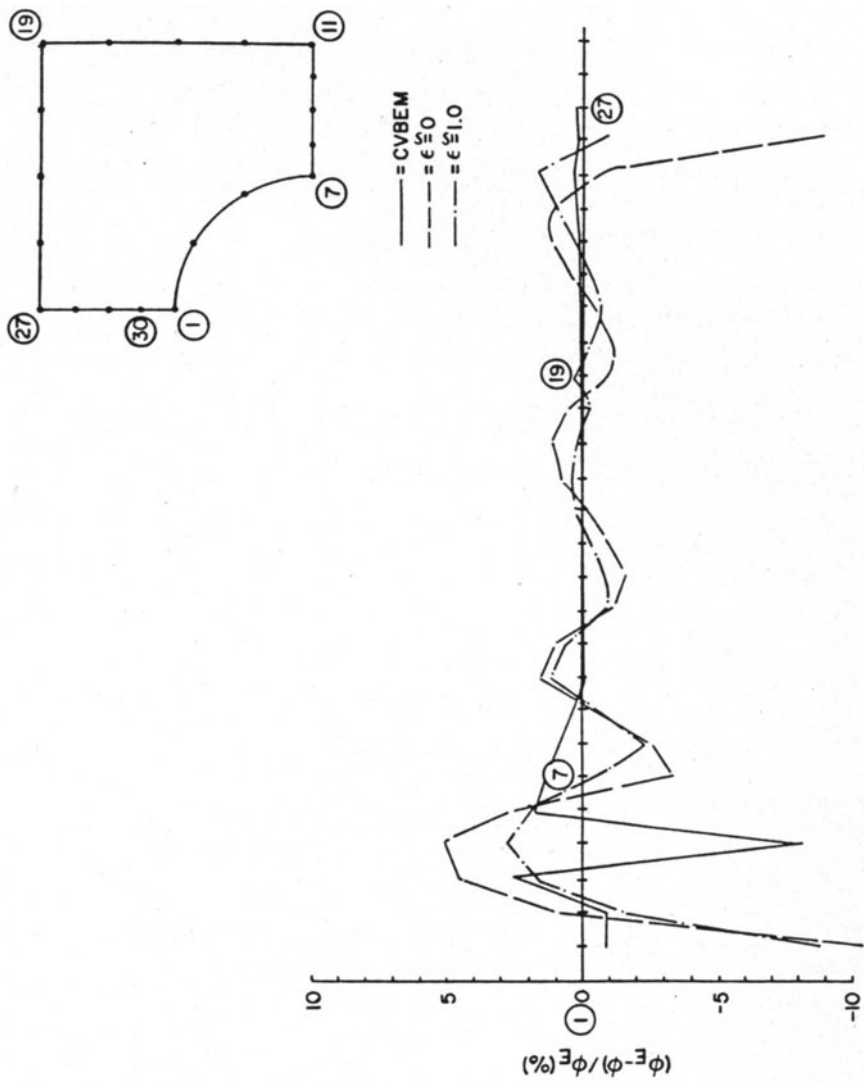


Fig. 6.7. Domain, Nodal Placement and Relative Error for Example 6.6

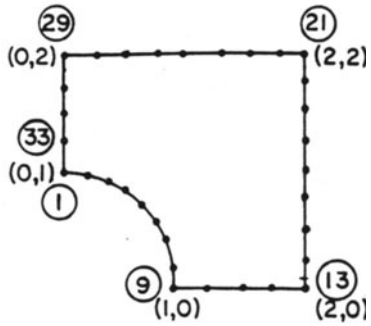


Fig. 6.8. Domain and Boundary Conditions for Example 6.7

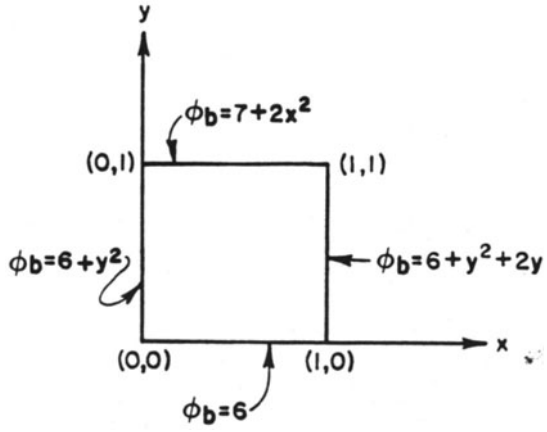


Fig. 6.10. Domain and Boundary Conditions for Example 6.9

If we included some singular terms, e. g. $\frac{1}{x}, \frac{1}{y}, \frac{1}{x^2}, \frac{1}{y^2}, \cdots$, into the set of basis function, the approximation function for the state variable $\hat{\phi}$ becomes

$$\hat{\phi} = 1.892x - 1.892y - 1.432x^2 + 1.432y^2 + .4409x^2y - .4409xy^2 - \frac{.434}{x} + \frac{.434}{y} + \frac{.1237}{x^2} - \frac{.1237}{y^2} - \frac{.01721}{x^3} + \frac{.01721}{y^3}$$

By increasing the boundary nodal points from 32 to 60 and the terms of the set of basis functions, we obtain respectively the approximation functions

$$\hat{\phi}_1 = 3.324x - 3.324y - 2.162x^2 + 2.162y^2 + .3721x^2y - .3721xy^2 + .6896x^3 - .6896y^3 - \frac{2.336}{x} + \frac{2.336}{y} + \frac{.9113}{x^2} - \frac{.9113}{y^2} - \frac{.1609}{x^3} + \frac{.1609}{y^3} + \frac{.01315}{x^4} - \frac{.01315}{y^4}$$

and

$$\hat{\phi}_2 = -.01232 + 3.905x - 3.937y + .3525xy - 2.892x^2 + 2.736y^2 + 2.686x^2y - 3.14xy^2 + 1.126x^3 - .895y^3 - 1.017x^3y + .299x^2y^2 + .9077xy^3 - .05475x^4 + .0745x^4y + .049x^3y^2 - .1336x^2y^3$$

The relative error $(\phi_{EXACT} - \hat{\phi})/\phi_{EXACT}$ of the approximation function $\hat{\phi}_1$ is smaller than the approximation function $\hat{\phi}_2$.

EXAMPLE 6.8: (Flow Net for Soil-Water Flow)

Figure 6.9 depicts the flow net for soil-water flow through a homogeneous soil as computed by the CVBEM. The L_2 approximation function of the state variable function $\hat{\phi}$ is given by

$$\hat{\phi} = 24. - .6549x - .0989y - 1.864xy - .06852x^2$$
$$+ 0.118y^2 + 2.4\times10^{-3}x^2y + 1.676\times10^{-3}xy^2 + 1.878\times10^{-3}x^3 - 3.129\times10^{-4}y^3$$

for $\varepsilon = 0.5$. Table 6.2 shows the interior nodal point values for both the CVBEM model and the L_2 approximation function.

Table 6.2. CVBEM and L_2 Computed Results for EXAMPLE 6.8

x	y	CVBEM	$\hat{\phi}$
4	4	19.9233	20.14
4	8	20.3408	20.35
4	12	20.9110	20.91
4	16	21.5258	21.70
4	20	22.1359	22.61
8	4	15.6543	15.34
8	8	16.5817	16.04
8	12	17.8253	17.30
8	16	19.0937	19.01
8	20	20.2656	21.04
12	4	10.7715	10.11
12	8	12.5019	11.6
12	12	14.7428	13.87
12	16	16.8199	16.8

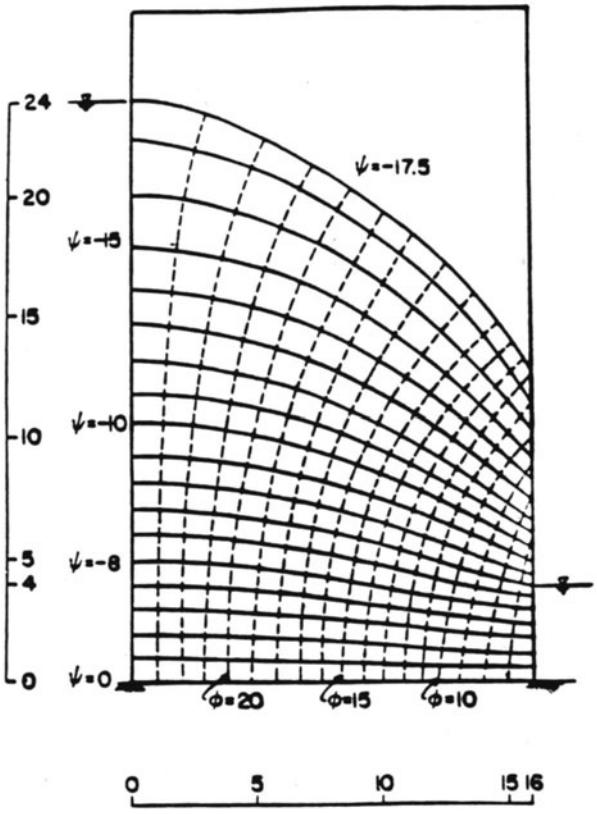


Fig. 6.9. Streamlines and Potentials for Soil-Water Flow Through a Homogeneous Soil for Example 6.8

6.3. Application to Other Linear Operators

The L_2 techniques can be applied to other linear operators. The example problems of Chapter 5 considers a linear integral equation, and two differential equation problems. This section considers nonhomogeneous problems involving potential theory.

EXAMPLE 6.9: (Potential Flow in an Anisotropic Domain)

Consider the linear operator, $\nabla^2\phi$,

$$\nabla^2\phi = T_{xx}\frac{\partial^2\phi}{\partial x^2} + T_{yy}\frac{\partial^2\phi}{\partial y^2}$$ on the unit square domain with boundary condition $\phi_b = 6 + y^2 + x^2y$ (Fig. 6.10).

The L_2 approximation functions for different T_{xx} and T_{yy} values are shown in Table 6.3.

Table 6.3. L_2 Function Coefficients and Relative Errors for EXAMPLE 6.9

T_{xx}	T_{yy}	1	x	y	xy	x^2	y^2	x^2y	xy^2	x^3	y^3	Maximum Relative error on Boundary	Maximum Relative error on Operator Relationship
1	1	5.856	.8996	.6616	.2092	-.8946	.8959	1.791	= 0	= 0	-.5967	10^{-2}	10^{-3}
2	1	5.878	.8277	.3158	.4275	-.8277	1.656	1.572	= 0	= 0	-1.048	10^{-2}	10^{-3}
1	2	5.849	.8657	.8637	.1007	-.8657	.4333	1.899	= 0	= 0	-.3165	10^{-2}	10^{-3}

EXAMPLE 6.10: (Poisson Problem)

Consider $\phi = 6 + y^2 + 2x^2y$ on the unit square domain with the linear operator $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$, such that

$$\nabla^2\phi = 4 + 2y$$

The L_2 approximation functions are listed in Table 6.4 for the various values of the weighting factor, ε.

Table 6.4. L_2 Solution of Poisson Problem for EXAMPLE 6.10

ε	$\hat{\phi}$
0	$x^2 + 2x^2y$ or $y^2 + 2x^2y$
$0<\varepsilon<1$	$6 + y^2 + 2x^2y$
$\varepsilon=1$	$6 + y^2 + 2x^2y$

EXAMPLE 6.11: (Poisson Problem in Anisotropic Domain)

Consider a linear operator, $\nabla^2\phi = T_{xx}\frac{\partial^2\phi}{\partial x^2} + T_{yy}\frac{\partial^2\phi}{\partial y^2}$, on the domain of EXAMPLE 6.10, we obtain

$$\nabla^2\phi = 4\,T_{xx} + 2\cdot T_{yy}\cdot y$$

The L_2 approximation functions for different values of T_{xx} and T_{yy} are listed in Table 6.5 for $\varepsilon = 0.5$.

Table 6.5. L_2 Approximation Results for EXAMPLE 6.11

T_{xx}	T_{yy}	$\hat{\phi}$
1	1	$6 + y^2 + 2x^2y$
2	1	$6.014 - 8.5\times10^{-2}x - .305y + .171xy + 8.5\times10^{-2}x^2 + 1.829y^2 + 1.829x^2y - .553y^3$
1	2	$5.993 + 3.37\times10^{-2}x + .174y - 6.75\times10^{-2}xy - 3.37\times10^{-2}x^2 + .517y^2 + 2.067x^2y + .3221y^3$
2	2	$6 + y^2 + 2x^2y$

The maximum relative error on the boundary is of magnitude 10^{-4} and on the interior is of magnitude 10^{-5} for the $T_{xx} \neq T_{yy}$.

EXAMPLE 6.12: (Poisson Problem in Triangular Domain)

Consider the EXAMPLE 6.10 on the problem domain of EXAMPLE 6.4 (Fig. 6.5), the L_2 approximation function provides the exact solution regardless the value of the weighting factor ($0<\varepsilon<1$).

EXAMPLE 6.13: (Poisson Problem with Transcendental Functions)

Consider $\phi = \sin x + yx + 10 + \cos^2 y$ on a unit square domain with a linear operator $\nabla^2\phi$, such that $\nabla^2\phi = 2 - 4\cos^2 y - \sin x$. The function ϕ can be expanded into an infinite series as

$$\phi = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$+ yx + 10$$

$$+ (1 + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{(2y)^{2k}}{(2k)!})$$

or

$$\phi = 11 + x + yx - \frac{x3}{3!} + \frac{x5}{5!} - \frac{x7}{7!} + \cdots$$

$$- y^2 + \frac{y4}{3} - \frac{2y6}{45} + \cdots$$

Table 6.6 lists the approximation functions for different nodal densities placements and basis function sets. The maximum relative error along the boundary is of the magnitude of order 10^{-4}. In all cases, $\varepsilon = 0.5$.

Table 6.6. L_2 Approximation Results for EXAMPLE 6.13

number of nodes Γ , Ω	$\hat{\phi}$
8 , 16	$11 + .999x + .049y + xy - .013x^2 - 1.29y^2$ $+ 0x^2y + 0xy^2 - .14x^3 + .53y^3$
8 , 25	$11 + .996x + .044y + xy - .011x^2 - 1.28y^2$ $+ 0x^2y + 0xy^2 - .14x^3 + .53y^3$
16 , 16	$11 + x + .047y + xy - .016x^2 - 1.29y^2$ $+ .9x10^{-3}x^2y - .64x10^{-4}xy^2 - .14x^3 + .53y^3$
16 , 25	$11 + .998x + .041y + xy - .013x^2 - 1.28y^2$ $- .86x10^{-4}x^2y - .43x10^{-5}xy^2 - .14x^3 + .53y^3$
16 , 25	$11 + .9992x + .0008654y + .9995xy$ $+ 6.07x10^{-3}x^2 - 1.07y^2 + 4.62x10^{-4}x^2y$ $+ 5.52x10^{-4}xy^2 - .1837x^3 + .1806y^3$ $- 7.82x10^{-12}x^3y - 5.48x10^{-4}x^2y^2$ $- 7.832x10^{-12}xy^3 + 1.986x^4 + 1.729y^4$

6.4. Computer Program: Two-Dimensional Potential Problems Using Real Variable Basis Functions

6.4.1. Introduction

The generalized Fourier series analysis program (GFSA1.FOR) consists of a main program and the subroutine FBASIS1.FOR. Figure 6.11 shows the simple flow chart of the GESA1 program. The subroutine FBASIS1 consists of the selected basis functions and their Laplacian relationship. The last section of subroutine FBASIS1 consists of the approximation function calculations for boundary values and the interior Laplacian relationship. The user can enter up to 20 basis functions. FORTRAN listings of the programs are included in the following section. The basis functions shown in subroutine FBASIS1 are 1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3. The provided Laplacian relationships are 0, 0, 0, 0, 2Txx, 2Tyy, 2Txx, 2xTyy, 2xTxx, 2yTyy according to $L(\phi) = Txx \frac{\partial^2\phi}{\partial x} + Tyy \frac{\partial^2\phi}{\partial y}$.

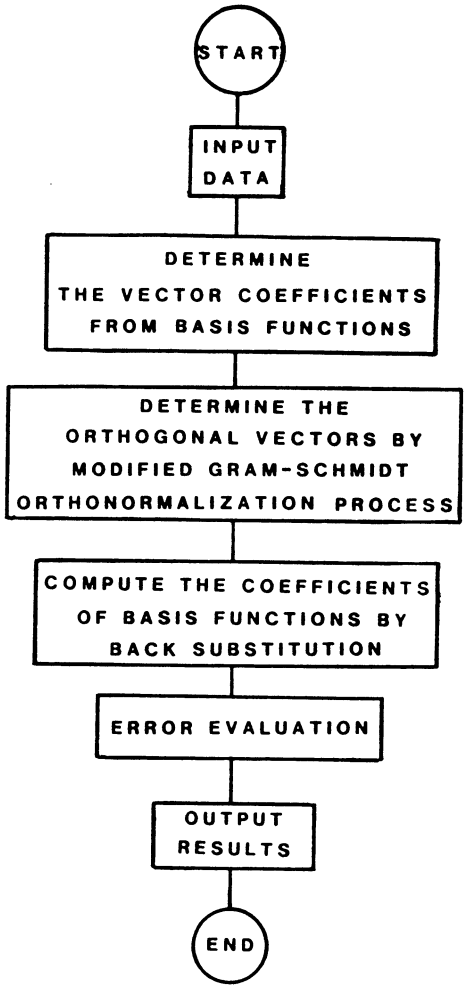


Fig. 6.11. Simple Flow Chart for Program GFSA1

6.4.2. Input Data Description

The input data are stored in data file--GFSA1.DAT. The input data should be entered sequentially as shown in Table 6.7. Table 6.8 shows the descriptions of input variables.

Table 6.7. Sequence of Input Data

Type of Data	Line Number	Variable
Control Data	1	NBON, NINT, EPSLO, KODE
	2	X(1), Y(1), VALUE(1)
Boundary Nodes	.	
	.	
	.	
	NBON+1	X(NBON), Y(NBON), VALUE(NBON)
Interior Nodes	NBON+2	X(NBON+1), Y(NBON+1), VALUE(NBON+1)
	.	
	.	
	.	
	NBON+1+NINT	X(NBON+NINT), Y(NBON+NINT), VALUE(NBON+NINT)

Table 6.8. Description of Input Variables

Variable	Descriptions
NBON	Total node number along domain boundary, Γ
NINT	Total node number on the domain, Ω
EPSLO	Weighting factor for inner product (between 0 and 1)
KODE	Output options 0 — summary of results 1 — detail of results
X(I)	Array stores x-coordinate for boundary and interior nodes
Y(I)	Array stores y-coordinate for boundary and interior nodes
VALUE(I)	Array stores boundary values and source term for interior nodal points.

6.4.3. Computer Program Listing

```
C
C       MAIN PROGRAM
C
C       THIS IS A GENERALIZED FOURIER SERIES ANALYSIS PROGRAM
C       WHICH SOLVES THE POISSON EQUATION
C
        IMPLICIT DOUBLE PRECISION(A-H,O-Z)
        COMMON/BLK 1/ X(200),Y(200)
        COMMON/BLK 2/ VALUE(200)
        COMMON/BLK 3/ B(20),S(20)
        COMMON/BLK 4/ G(20,200)
        COMMON/BLK 5/ XK(20,20)
C
C       OPEN DATA FILES
C
C.......INPUT DATA FILE  = GFSA.DAT
C.......OUTPUT DATA FILE = GFSA.ANS
C
        NRD=1
        NWT=2
        OPEN(UNIT=NRD,FILE='GFSA.DAT',STATUS='OLD')
        OPEN(UNIT=NWT,FILE='GFSA.ANS',STATUS='NEW')
C
C       READ INPUT DATA
C
C.......DEFINITION OF VARIABLES
C
C       NBON : TOTAL NODE NUMBER ON THE BOUNDARY
C       NINT : TOTAL NODE NUMBER ON THE INTERIOR
C       EPSLO: WEIGHTING FACTOR FOR THE INNER PRODUCT
C       KODE : 0 - SUMMARY OF RESULTS
C              1 - DETAIL OF RESULTS
C       X(I), Y(I) : X- AND Y- COORDINATES FOR NODE I
C       VALUE(I)   : BOUNDARY VALUE OR VALUE OF SOURCE TERM FOR NODE I
C
        READ(NRD,*) NBON,NINT,EPSLO,KODE
        DO 7 I=1,NBON
7       READ(NRD,*) X(I),Y(I),VALUE(I)
        WRITE(NWT,2)
        WRITE(NWT,17)EPSLO
C
C       OUTPUT FORMATS
C
2       FORMAT(//,10X,'*** INFORMATION OF BOUNDARY NODES ***',
     1  //,' NODE   X(I)      Y(I)        VALUE(I)')
3       FORMAT(2X,2I3,3X,D10.4)
4       FORMAT(/,10X,'*** APPROXIMATE INTERIOR NODAL VALUES AND ERRORS'
     1  ,' ***',//,6X,'NODE',9X,'EXACT',9X,'APPROXIMATION',5X,
     2  'RELATIVE ERROR',8X,'F(X,Y)',6X,'ESTIMATED F(X,Y)',3X,
     3  'RELATIVE ERROR')
5       FORMAT(3X,I5,3(8X,D10.4))
8       FORMAT(1X,I3,2X,F8.3,2X,F8.3,4X,F7.2)
17      FORMAT(/,'WEIGHTING FACTOR FOR INNER PRODUCT = ',F4.2/)
22      FORMAT(10(1X,D10.4))
```

```
 23      FORMAT(/,10X,'*** NODE # ',I3,' ***')
 24      FORMAT(/,10X,'*** NODAL POINT VECTOR EXPANSION, F(I) ***'/)
 25      FORMAT(/,10X,'*** ORTHOGONAL VECTOR EXPANSION, G(I) ***'/)
 26      FORMAT(/,10X,'*** ORTHOGONAL TEST (G(I),G(J)) ***',//,
     1   '   I  J  (G(I),G(J))')
 28      FORMAT(/,10X,'EVALUATION COEFFICIENTS',
     1   ' (G(I),B(I))/(G(I),G(I)) ***'/)
 29      FORMAT(/,10X,'*** BACK SUBSTITUTION COEFFICIENTS ***'/)
 36      FORMAT(/,2X,'*** BESSEL INEQUALITY ***',/,2X,D10.4,
     1   ' >= ',D10.4,' AND THE DIFFERENCE IS ',D10.4,/,120('-')/)
 38      FORMAT(2(1X,F10.4),1X,I2,3(1X,F10.4))
 41      FORMAT(//,10X,'*** INFORMATION OF INTERIOR POINTS ***',
     1   //,' POINT  X(I)      Y(I)        VALUE(I)')
 45      FORMAT(/,10X,'*** APPROXIMATE BOUNDARY VALUES AND ERRORS ***',
     1   //,5X,'POINT',9X,'EXACT',9X,'APPROXIMATION',5X,'RELATIVE',
     1   ' ERROR'/)
 57      FORMAT(3X,I5,6(8X,D10.4))
 81      FORMAT(1X,I3,2X,F8.3,2X,F8.3,4X,F7.2)
 99      FORMAT(/,120('='))
C
C       INITIALIZE CONSTANTS
C
        NTOL=NBON
        SAREA=0.
        EX=0.
C
C       READ BOUNDARY INFORMATION FOR BOUNDARY NODES
C
        DO 300 I=1,NBON
        WRITE(NWT,8) I,X(I),Y(I),VALUE(I)
        SAREA=SAREA+EPSLO*VALUE(I)*VALUE(I)
300     CONTINUE
C
C       READ X-COORDINATE, Y-COORDINATE, AND THE SOURCE TERM
C       FOR THE INTERIOR POINTS
C
        WRITE(NWT,41)
        DO 305 I=1,NINT
        NTOL=NTOL+1
        READ (NRD,*) X(NTOL),Y(NTOL),VALUE(NTOL)
        SAREA=SAREA+VALUE(NTOL)*VALUE(NTOL)*(1.-EPSLO)
        WRITE(NWT,81)NTOL,X(NTOL),Y(NTOL),VALUE(NTOL)
305     CONTINUE
C
C       DETERMINE THE VECTOR COFFICIENTS FROM BASIS FUNCTIONS
C
        IF(KODE.EQ.1)WRITE(NWT,99)
        IF(KODE.EQ.1)WRITE(NWT,24)
        DO 315 I=1,NTOL
        XX=X(I)
        YY=Y(I)
        CALL BASIS(NBAS,XX,YY,I,NBON,VBAR,FBAR,0)
315     CONTINUE
```

```
      IF(KODE.NE.1)GOTO 321
      DO 319 I=1,NBAS
      WRITE(NWT,22)(G(I,J),J=1,NTOL)
319   CONTINUE
C
C     USE THE MODIFIED GRAM-SCHMIDT ORTHONORMALIZATION PROCESS
C     TO DETERMINE SERIES OF ORTHOGONAL VECTORS
C
321   DO 320 I=1,NBAS
      SUM=0.
      DO 323 J=1,NTOL
      IF(J.LE.NBON)SUM=SUM+EPSLO*G(I,J)*G(I,J)
      IF(J.GT.NBON)SUM=SUM+(1.-EPSLO)*G(I,J)*G(I,J)
323   CONTINUE
      SUM=SQRT(SUM)
      S(I)=SUM
      IF(SUM.LT..0000001)S(I)=0.
      DO 325 J=1,NTOL
      IF(SUM.GT.0.)G(I,J)=G(I,J)/SUM
      IF(SUM.EQ.0.)G(I,J)=0.
325   CONTINUE
      IF(I.EQ.NBAS)GOTO 320
      IP1=I+1
      DO 310 KK=IP1,NBAS
      SUM1=0.
      DO 330 J=1,NTOL
      IF(J.GT.NBON)GOTO 335
      SUM1=SUM1+G(I,J)*G(KK,J)*EPSLO
      GO TO 330
335   SUM1=SUM1+(1.-EPSLO)*G(KK,J)*G(I,J)
330   CONTINUE
      XXK=-1.*SUM1
      IF(S(I).GT.0.)XK(KK,I)=XXK/S(I)
      IF(S(I).EQ.0.)XK(KK,I)=0.
      DO 340 J=1,NTOL
      G(KK,J)=G(KK,J)+XXK*G(I,J)
340   CONTINUE
310   CONTINUE
320   CONTINUE
      IF(KODE.NE.1.)GO TO 55
      WRITE(NWT,25)
      DO 47 I=1,NBAS
47    WRITE(NWT,22)(G(I,J),J=1,NTOL)
C.....CHECK ORTHOGONALITY OF VECTORS G(I)
      WRITE(NWT,26)
55    DO 50 I=1,NBAS
      IP1=I+1
      IF(I.EQ.NBAS)GO TO 80
      DO 70 K=IP1,NBAS
      SUM=0.
      DO 60 J=1,NTOL
      IF(J.LE.NBON)SUM=SUM+G(I,J)*G(K,J)*EPSLO
      IF(J.GT.NBON)SUM=SUM+G(I,J)*G(K,J)*(1.-EPSLO)
60    CONTINUE
```

```
      IF(KODE.EQ.1)WRITE(NWT,3)I,K,SUM
70    CONTINUE
50    CONTINUE
80    CONTINUE
C.....COMPUTE THE CONFFICIENTS OF B(I)=(VALUE(I),G(I))
      WRITE(NWT,28)
      SUM=0.
      DO 120 I=1,NBAS
      BK1=0.
      DO 130 J=1,NTOL
      IF(J.LE.NBON)BK1=BK1+VALUE(J)*G(I,J)*EPSLO
      IF(J.GT.NBON)BK1=BK1+VALUE(J)*G(I,J)*(1.-EPSLO)
130   CONTINUE
C.....COMPUTE THE NORM OF THE GENERALIZED FOURIER COEFFICIENTS
      IF(S(I).GT.0.)B(I)=BK1/S(I)
      IF(S(I).EQ.0.)B(I)=0.
      SUM=SUM+BK1*BK1
120   CONTINUE
      WRITE(NWT,22)(B(I),I=1,NBAS)
C.....COMPUTE BASIS FUNCTION COEFFICIENTS (BACK-SUBSTITUTION)
660   DO 200 I=NBAS,1,-1
      IF(I.EQ.NBAS)XK(NBAS,I)=B(NBAS)
      IF(I.NE.NBAS)XK(NBAS,I)=XK(NBAS,I)*B(NBAS)+B(I)
200   CONTINUE
      NTOT1=NBAS-1
      DO 210 I=NTOT1,1,-1
      DO 210 J=I,1,-1
      IF(I.EQ.J)GO TO 210
      IF(I.NE.J)XK(NBAS,J)=XK(NBAS,I)*XK(I,J)+XK(NBAS,J)
210   CONTINUE
      WRITE(NWT,29)
      WRITE(NWT,22)(XK(NBAS,I),I=1,NBAS)
C
C     BOUNDARY AND INTERIOR VALUES APPROXIMATION
C
C.....APPROXIMATE BOUNDARY POINT VALUES
      WRITE(NWT,99)
      WRITE(NWT,45)
      DO 470 I=1,NBON
      XX=X(I)
      YY=Y(I)
      CALL BASIS(NBAS,XX,YY,I,NBON,VBAR,FBAR,1)
      IF(VALUE(I).EQ.0.)XD=-9999.
      IF(VALUE(I).NE.0.)XD=(VALUE(I)-VBAR)/VALUE(I)
      WRITE(NWT,5)I,VALUE(I),VBAR,XD
470   CONTINUE
C.....APPROXIMATE INTERIOR POINT VALUES
      WRITE(NWT,99)
      WRITE(NWT,4)
      DO 280 I=NBON+1,NTOL
      CALL BASIS(NBAS,X(I),Y(I),I,NBON,VBAR,FBAR,1)
      EX=0.
```

```
      IF(EX.NE.0.)XD=(EX-VBAR)/EX
      IF(EX.EQ.0.)XD=-9999.
      IF(VALUE(I).EQ.0.)FD=-9999.
      IF(VALUE(I).NE.0.)FD=(VALUE(I)-FBAR)/VALUE(I)
      WRITE(NWT,57)I,EX,VBAR,XD,VALUE(I),FBAR,FD
280   CONTINUE
      WRITE(NWT,99)
C.....BESSEL'S INEQUALITY
      DIFF=SAREA-SUM
      IF(ABS(DIFF).LT.0.00001)DIFF=0.
      WRITE(NWT,36)SAREA,SUM,DIFF
      STOP
      END
```

```
C*********************************************************************
      SUBROUTINE BASIS(NB,XX,YY,I,NNOD,VBAR,FBAR,KK)
C
C     THIS SUBROUTINE EVALUATES THE BASIS FUNCTION VALUES AND
C     THE LAPLACIAN OF THE BASIS FUNCTION VALUES
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/BLK 4/ G(20,200)
      COMMON/BLK 5/ XK(20,20)
      NB=10
      TXX=1.
      TYY=1.
      X2=XX*XX
      Y2=YY*YY
      XY=XX*YY
      IF(KK.EQ.1)GOTO 200
      IF(I.GT.NNOD)GOTO 100
C.....EVALUTE BASIS FUNCTIONS
      G(1,I)=1.
      G(2,I)=XX
      G(3,I)=YY
      G(4,I)=XY
      G(5,I)=X2
      G(6,I)=Y2
      G(7,I)=X2*YY
      G(8,I)=XX*Y2
      G(9,I)=XX*X2
      G(10,I)=YY*Y2
      RETURN
C.....EVALUTE LAPLACIAN OF BASIS FUNCTION
100   G(1,I)=0.
      G(2,I)=0.
      G(3,I)=0.
      G(4,I)=0.
      G(5,I)=2.*TXX
      G(6,I)=2.*TYY
      G(7,I)=2.*YY*TXX
      G(8,I)=2.*XX*TYY
      G(9,I)=6.*XX*TXX
      G(10,I)=6.*YY*TYY
      RETURN
C.....EVALUTE APPROXIMATION FUNCTION (VBAR)
C.....AND LAPLACIAN OF APPROXIMTION FUNCTION (FBAR)
200   VBAR=XK(NB,1)+XK(NB,2)*XX+XK(NB,3)*YY+XK(NB,4)*XY+XK(NB,5)*X2
     1  +XK(NB,6)*Y2+XK(NB,7)*X2*YY+XK(NB,8)*XX*Y2+XK(NB,9)*XX*X2
     1  +XK(NB,10)*YY*Y2
      FBAR=2.*XK(NB,5)*TXX+2.*XK(NB,7)*YY*TXX
     1  +6.*XK(NB,9)*XX*TXX
     1  +2.*XK(NB,6)*TYY+2.*XK(NB,8)*XX*TYY
     1  +6.*XK(NB,10)*YY*TYY
      RETURN
      END
```

EXAMPLE 6.14: (Ideal Fluid Flow Program Input)

The potential function for Ideal fluid flow around a 90° corner is studied in this example. (see Example 6.3)

Input Data

```
16 16 .5 0
0.00 0.00 0.00
0.25 0.00 0.0625
0.50 0.00 0.25
0.75 0.00 0.5625
1.00 0.00 1.00
1.00 0.25 0.9375
1.00 0.50 0.75
1.00 0.75 0.4375
1.00 1.00 0.00
0.75 1.00 -.4375
0.50 1.00 -.75
0.25 1.00 -.9375
0.00 1.00 -1.0
0.00 0.75 -.5625
0.00 0.50 -.25
0.00 0.25 -.0625
0.20 0.20 0.00
0.20 0.40 0.00
0.20 0.60 0.00
0.20 0.80 0.00
0.40 0.20 0.00
0.40 0.40 0.00
0.40 0.60 0.00
0.40 0.80 0.00
0.60 0.20 0.00
0.60 0.40 0.00
0.60 0.60 0.00
0.60 0.80 0.00
0.80 0.20 0.00
0.80 0.40 0.00
0.80 0.60 0.00
0.80 0.80 0.00
```

Summary Results

*** INFORMATION OF BOUNDARY NODES ***

NODE	X(I)	Y(I)	VALUE(I)
WEIGHTING FACTOR FOR INNER PRODUCT = .50			
1	.000	.000	.00
2	.250	.000	.06
3	.500	.000	.25
4	.750	.000	.56
5	1.000	.000	1.00
6	1.000	.250	.94
7	1.000	.500	.75
8	1.000	.750	.44
9	1.000	1.000	.00
10	.750	1.000	-.44
11	.500	1.000	-.75
12	.250	1.000	-.94
13	.000	1.000	-1.00
14	.000	.750	-.56
15	.000	.500	-.25
16	.000	.250	-.06

*** INFORMATION OF INTERIOR POINTS ***

POINT	X(I)	Y(I)	VALUE(I)
17	.200	.200	.00
18	.200	.400	.00
19	.200	.600	.00
20	.200	.800	.00
21	.400	.200	.00
22	.400	.400	.00
23	.400	.600	.00
24	.400	.800	.00
25	.600	.200	.00
26	.600	.400	.00
27	.600	.600	.00
28	.600	.800	.00
29	.800	.200	.00
30	.800	.400	.00
31	.800	.600	.00
32	.800	.800	.00

EVALUATION COEFFICIENTS (G(I),B(I))/(G(I),G(I)) ***

.1472D-16 .1000D+01 -.1000D+01 .1550D-14 .4140D-02 -.1000D+01 -.1042D-15 -.1644D-15 .5267D-14 .2448D-14

*** BACK SUBSTITUTION COEFFICIENTS ***

.1855D-14 -.1881D-14 -.1031D-13 .2464D-13 .1000D+01 -.1000D+01 -.7373D-14 -.1581D-13 .5267D-14 .2448D-14

*** APPROXIMATE BOUNDARY VALUES AND ERRORS ***

POINT	EXACT	APPROXIMATION	RELATIVE ERROR
1	.0000D+00	.1855D-14	-.9999D+04
2	.6250D-01	.6250D-01	-.1754D-13
3	.2500D+00	.2500D+00	-.2220D-15
4	.5625D+00	.5625D+00	.1184D-14
5	.1000D+01	.1000D+01	.7772D-15
6	.9375D+00	.9375D+00	-.4737D-15
7	.7500D+00	.7500D+00	-.7401D-15
8	.4375D+00	.4375D+00	-.1269D-15
9	.0000D+00	-.1053D-14	-.9999D+04
10	-.4375D+00	-.4375D+00	.0000D+00
11	-.7500D+00	-.7500D+00	.1184D-14
12	-.9375D+00	-.9375D+00	.1184D-14
13	-.1000D+01	-.1000D+01	.1110D-15
14	-.5625D+00	-.5625D+00	-.2566D-14
15	-.2500D+00	-.2500D+00	-.5773D-14
16	-.6250D-01	-.6250D-01	-.4885D-14

*** APPROXIMATE INTERIOR NODAL VALUES AND ERRORS ***

NODE	EXACT	APPROXIMATION	RELATIVE ERROR	F(X,Y)	ESTIMATED F(X,Y)	RELATIVE ERROR
17	.0000D+00	.2832D-15	-.9999D+04	.0000D+00	.2083D-15	-.9999D+04
18	-.1200D+00	-.1200D+00	-.3123D-14	.0000D+00	.1962D-15	-.9999D+04
19	-.3200D+00	-.3200D+00	-.1735D-14	.0000D+00	.1843D-15	-.9999D+04
20	-.6000D+00	-.6000D+00	-.1850D-15	.0000D+00	.1723D-15	-.9999D+04
21	.1200D+00	.1200D+00	-.1503D-14	.0000D+00	.2065D-15	-.9999D+04
22	.0000D+00	-.5138D-16	-.9999D+04	.0000D+00	.1945D-15	-.9999D+04
23	-.2000D+00	-.2000D+00	-.2776D-15	.0000D+00	.1826D-15	-.9999D+04
24	-.4800D+00	-.4800D+00	.4626D-15	.0000D+00	.1705D-15	-.9999D+04
25	.3200D+00	.3200D+00	.0000D+00	.0000D+00	.2046D-15	-.9999D+04
26	.2000D+00	.2000D+00	-.2776D-15	.0000D+00	.1926D-15	-.9999D+04
27	.0000D+00	.1092D-15	-.9999D+04	.0000D+00	.1807D-15	-.9999D+04
28	-.2800D+00	-.2800D+00	.7930D-15	.0000D+00	.1687D-15	-.9999D+04
29	.6000D+00	.6000D+00	.1850D-15	.0000D+00	.2027D-15	-.9999D+04
30	.4800D+00	.4800D+00	-.4626D-15	.0000D+00	.1907D-15	-.9999D+04
31	.2800D+00	.2800D+00	-.7930D-15	.0000D+00	.1789D-15	-.9999D+04
32	.0000D+00	.2268D-16	-.9999D+04	.0000D+00	.1669D-15	-.9999D+04

*** BESSEL INEQUALITY ***
.3016D+01 >= .3016D+01 AND THE DIFFERENCE IS .0000D+00

In this example, since we know the exact solution of the problem. Additional computer statements are included in the program to evaluate the relative error for the Laplacian relationship of the interior points.

EXAMPLE 6.15

The soil-water flow through a homogeneous soil is studied in this example. (see Example 6.8)

Input Data

```
17 14 .5 0
0. 0. 24.0
5. 0. 18.75
11. 0. 11.56
16. 0. 4.0
16. 4. 4.0
16. 8. 8.0
16. 12.62 12.62
14. 17.75 17.75
12. 19.86 19.86
9. 21.71 21.71
6. 23.02 23.02
3. 23.5 24.
0. 24. 24.
0. 20. 24.
0. 15. 24.
0. 10. 24.
0. 5. 24.
4. 4. 0.
4. 8. 0.
4. 12. 0.
4. 16. 0.
4. 20. 0.
8. 4. 0.
8. 8. 0.
8. 12. 0.
8. 16. 0.
8. 20. 0.
12. 4. 0.
12. 8. 0.
12. 12. 0.
12. 16. 0.
```

Summary Results

*** INFORMATION OF BOUNDARY NODES ***

NODE	X(I)	Y(I)	VALUE(I)
WEIGHTING FACTOR FOR INNER PRODUCT = .50			
1	.000	.000	24.00
2	5.000	.000	18.75
3	11.000	.000	11.56
4	16.000	.000	4.00
5	16.000	4.000	4.00
6	16.000	8.000	8.00
7	16.000	12.620	12.62
8	14.000	17.750	17.75
9	12.000	19.860	19.86
10	9.000	21.710	21.71
11	6.000	23.020	23.02
12	3.000	23.500	24.00
13	.000	24.000	24.00
14	.000	20.000	24.00
15	.000	15.000	24.00
16	.000	10.000	24.00
17	.000	5.000	24.00

*** INFORMATION OF INTERIOR POINTS ***

POINT	X(I)	Y(I)	VALUE(I)
18	4.000	4.000	.00
19	4.000	8.000	.00
20	4.000	12.000	.00
21	4.000	16.000	.00
22	4.000	20.000	.00
23	8.000	4.000	.00
24	8.000	8.000	.00
25	8.000	12.000	.00
26	8.000	16.000	.00
27	8.000	20.000	.00
28	12.000	4.000	.00
29	12.000	8.000	.00
30	12.000	12.000	.00
31	12.000	16.000	.00

*** EVALUATION COEFFICIENTS (G(I),B(I))/(G(I),G(I)) ***

.1819D+02 -.9360D+00 .2876D+00 .4784D-01 -.2360D-01 .2178D-02 .1155D-02 .1484D-02 .1990D-02 -.3129D-03

*** BACK SUBSTITUTION COEFFICIENTS ***

.2400D+02 -.6549D+00 -.9890D-01 -.1864D-01 -.6852D-01 .1180D-01 .2409D-02 .1676D-02 .1878D-02 -.3129D-03

*** APPROXIMATE BOUNDARY VALUES AND ERRORS ***

POINT	EXACT	APPROXIMATION	RELATIVE ERROR
1	.2400D+02	.2400D+02	.6667D-04
2	.1875D+02	.1925D+02	-.2644D-01
3	.1156D+02	.1100D+02	.4813D-01
4	.4000D+01	.3673D+01	.8185D-01
5	.4000D+01	.5149D+01	-.2874D+00
6	.8000D+01	.7742D+01	.3222D-01
7	.1262D+02	.1197D+02	.5163D-01
8	.1775D+02	.1791D+02	-.9201D-02
9	.1986D+02	.2014D+02	-.1424D-01
10	.2171D+02	.2185D+02	-.6226D-02
11	.2302D+02	.2292D+02	.4224D-02
12	.2400D+02	.2358D+02	.1770D-01
13	.2400D+02	.2410D+02	-.4130D-02
14	.2400D+02	.2424D+02	-.9970D-02
15	.2400D+02	.2412D+02	-.4792D-02
16	.2400D+02	.2388D+02	.5125D-02
17	.2400D+02	.2376D+02	.1000D-01

*** APPROXIMATE INTERIOR NODAL VALUES AND ERRORS ***

NODE	EXACT	APPROXIMATION	RELATIVE ERROR	F(X,Y)	ESTIMATED F(X,Y)	RELATIVE ERROR
18	.0000D+00	.2014D+02	-.9999D+04	.0000D+00	-.4318D-01	-.9999D+04
19	.0000D+00	.2035D+02	-.9999D+04	.0000D+00	-.3141D-01	-.9999D+04
20	.0000D+00	.2091D+02	-.9999D+04	.0000D+00	-.1964D-01	-.9999D+04
21	.0000D+00	.2170D+02	-.9999D+04	.0000D+00	-.7876D-02	-.9999D+04
22	.0000D+00	.2261D+02	-.9999D+04	.0000D+00	.3890D-02	-.9999D+04
23	.0000D+00	.1534D+02	-.9999D+04	.0000D+00	.1532D-01	-.9999D+04
24	.0000D+00	.1604D+02	-.9999D+04	.0000D+00	.2708D-01	-.9999D+04
25	.0000D+00	.1730D+02	-.9999D+04	.0000D+00	.3885D-01	-.9999D+04
26	.0000D+00	.1901D+02	-.9999D+04	.0000D+00	.5062D-01	-.9999D+04
27	.0000D+00	.2104D+02	-.9999D+04	.0000D+00	.6238D-01	-.9999D+04
28	.0000D+00	.1011D+02	-.9999D+04	.0000D+00	.7381D-01	-.9999D+04
29	.0000D+00	.1160D+02	-.9999D+04	.0000D+00	.8558D-01	-.9999D+04
30	.0000D+00	.1387D+02	-.9999D+04	.0000D+00	.9734D-01	-.9999D+04
31	.0000D+00	.1680D+02	-.9999D+04	.0000D+00	.1091D+00	-.9999D+04

*** BESSEL INEQUALITY ***
.3242D+04 >= .3240D+04 AND THE DIFFERENCE IS .1494D+01

The relative error is defined as:

$$\text{Relative Error} = \begin{cases} \dfrac{|\ \text{Exact} - \text{Approximation}\ |}{\text{Exact}} & ;\ \text{Exact} \neq 0 \\ -999 & ;\ \text{otherwise} \end{cases}$$

CHAPTER 7

COUPLING THE BEST APPROXIMATION AND COMPLEX VARIABLE BOUNDARY ELEMENT METHODS

7.0. Introduction

From (5.4), a linear operation equation is solved by the Best Approximation Method by use of the inner product

$$(u,v) = \int_{\Gamma} uv\,d\Gamma + \int_{\Omega} Lu\ Lv\ d\Omega \tag{7.1}$$

where the integration over Γ includes both the spatial and temporal boundary conditions (i.e., initial conditions in a diffusion problem).

By the clever choice of basis functions, the inner product can be simplified. For example, choosing basis functions which satisfy the operator relationship ($Lu = Lv = 0$ in (7.1)) reduces (7.1) to

$$(u,v) = \int_{\Gamma} uv\,d\Gamma \tag{7.2}$$

In this chapter, the two-dimensional Laplace equation (or Poisson equation) is restudied by using the Best Approximation Method where the set of basis functions are analytic. That is, an approximation function is developed which is the sum of complex variable analytic functions. Because only the boundary component of (7.1) is used in (7.2), the resulting approximator is a variant of the Complex Variable Boundary Element Method (Hromadka, 1984).

7.1. The Complex Variable Boundary Element Method

7.1.1. Objectives

The objective in using the Complex Variable Boundary Element Method (or CVBEM) is to approximate analytic complex functions. More specifically, if ω is a two-dimensional complex function which is analytic over a simply connected domain Ω with boundary values $\omega(\zeta)$ for $\zeta \in \Gamma$ (Γ is a simple closed contour), then the real (ϕ) and imaginary (ψ) parts of $\omega = \phi + i\psi$ both satisfy the Laplace equation over Ω. Thus, two-dimensional potential problems can be solved numerically by the CVBEM.

The development of the CVBEM for engineering applications is detailed in Hromadka (1984). Generally speaking, the CVBEM is a boundary integral technique and, consequently, a literature review of this class of numerical methods can be found in other books such as Lapidus and Pinder (1982).

However in this chapter, the CVBEM departs from the other boundary integral methods by using a best approximation in satisfying boundary conditions. Instead of developing a square order m matrix system for m boundary nodes by collocating the boundary integral equation at nodal point boundary values, the CVBEM is now expanded as a generalized Fourier series--eliminating the matrix solution entirely. Boundary conditions are approximated in a "mean-square" error sense in that a new vector space norm is defined which is analogous to the L_2 norm, and then minimized by the selection of complex coefficients to be associated to each nodal point located on the problem boundary, Γ.

7.1.2. Definition 7.1: (Working Space, W_Ω)

Let Ω be a simply connected convex domain with a simple closed piecewise linear boundary Γ and with its centroid located at 0 + 0i. Then $\omega \in W_\Omega$ has the properties

(i) $\omega(z)$ is analytic over Ω

(ii) $\lim_{\delta \to 1} \int_\Gamma |\omega(\delta\zeta)|^2 d\Gamma \leq M < \infty$

7.1.3. Definition 7.2: (the Function $||\omega||$)

A key element in the CVBEM development of this chapter is the definition of a norm and inner-product. In the following sections, insight into the new norm function is presented by an analogy to the well known $L_2(\Gamma)$ norm and inner-product.

The symbol $||\omega||$ for $\omega \in W_\Omega$ is notation for

$$||\omega|| = \left[\int_{\Gamma_\phi} (Re\omega)^2 \, d\mu + \int_{\Gamma_\psi} (Im\omega)^2 \, d\mu \right]^{\frac{1}{2}}$$

The symbol $||\omega||_p$ for $\omega \in W_\Omega$ is notation for

$$||\omega||_p = \left[\int_\Gamma |\omega(\zeta)|^p d\mu \right]^{1/p}, \quad p \geq 1$$

Of importance is the case of p = 2:

$$||\omega||_2 = \left[\int_\Gamma |\omega(\zeta)|^2 d\mu \right]^{\frac{1}{2}} = \left[\int_\Gamma \left(\left[Re\omega \right]^2 + \left[Im\omega \right]^2 \right) d\mu \right]^{\frac{1}{2}}$$

7.1.4. Almost Everywhere (ae) Equality

Because sets of Lebesque measure zero have no effect on integration, almost-everywhere (ae) equality on Γ indicates the same class of element. Thus for $\omega \in W_\Omega$, $[\omega] = \{\omega \in W_\Omega : \omega(\zeta)$ are equal ae for $\zeta \in \Gamma\}$. For example, $[0] = \{\omega \in W_\Omega : \omega(\zeta) = 0$ ae, $\zeta \in \Gamma\}$. When understood, the notation "[ ]" will be dropped.

7.1.5. Theorem (relationship of $||\omega||$ to $||\omega||_2$)

Let $\omega \in W_\Omega$. Then $||\omega||_2^2 = ||\omega||^2 + ||i\omega||^2$

Proof

Let $\omega = \phi + i\psi$.

Then $$||\omega||_2^2 = \int_\Gamma |\omega(\zeta)|^2 d\mu$$

$$= \int_\Gamma (\phi^2 + \psi^2) d\mu$$

$$= \int_{\Gamma_\phi} \phi^2 d\mu + \int_{\Gamma_\psi} \phi^2 d\mu + \int_{\Gamma_\phi} \psi^2 d\mu + \int_{\Gamma_\psi} \psi^2 d\mu$$

$$= ||\omega||^2 + \int_{\Gamma_\phi} (-\psi)^2 d\mu + \int_{\Gamma_\psi} \phi^2 d\mu$$

$$= ||\omega||^2 + ||i\omega||^2$$

7.1.6. Theorem

Let $\omega \in W_\Omega$. Then $||\omega||_2 = 0 \Rightarrow ||\omega|| = 0$.

Proof

$||\omega||_2 = 0$ implies $||\omega||^2 = ||\omega|| + ||i\omega||^2 = 0$.

7.1.7. Theorem

Let $\omega \in W_\Omega$. Then $||\omega|| \leq ||\omega||_2$.

Proof

Let $\omega = \phi + i\psi$. Then

$$||\omega||_2^2 = \int_\Gamma |\omega(\zeta)|^2 d\mu = \int_\Gamma \phi^2 d\mu + \int_\Gamma \psi^2 d\mu$$

$$= ||\omega||^2 + ||i\omega||^2$$

Because $||i\omega||^2 \geq 0$, then $||\omega||^2 \leq ||\omega||_2^2$.

7.2. Mathematical Development

7.2.1. Discussion: (A Note on Hardy Spaces)

The H^p spaces (or Hardy spaces) are well documented in the literature (e.g. Duren, 1970). Of special interest are the $E^p(\Omega)$ spaces of complex valued functions. If $\omega \in E^2(\Omega)$, then ω satisfies the conditions of Definition 1.3 for W_Ω, where $||\omega(\delta\zeta)||_2$ is bounded as $\delta \to 1$. Finally, if $\omega \in E^2(\Omega)$ then the Cauchy integral representation of $\omega(z)$ for $z \in \Omega$ applies. It is seen that $W_\Omega \subset E^2(\Omega)$.

7.2.2. Theorem (Boundary Integral Representation)

Let $\omega \in W_\Omega$ and $z \in \Omega$. Then

$$\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta)d\zeta}{\zeta - z}$$

Proof

For $\omega \in W_\Omega$, then $\omega \in E^2(\Omega)$ and the result follows immediately.

7.2.3. Almost Everywhere (ae) Equivalence

For $\omega \in W_\Omega$, functions $x \in W_\Omega$ which are equal to ω ae on Γ represent an equivalence class of functions which may be noted as $[\omega]$. Thus for x and y in W_Ω, $x = y$ implies $x = y$ over Ω and $\int_\Gamma |x - y| d\mu = 0$.

For simplicity, $\omega \in W_\Omega$ is understood to indicate $[\omega]$. This follows directly from the boundary integral representation of $\omega(z)$ for $z \in \Omega$, and the fact that integrals over sets of measure zero have no effect on the integral value.

7.2.4. Theorem (Uniqueness of Zero Element in W_Ω)

Let $\omega \in W_\Omega$ and $\phi = 0$ ae on Γ_ϕ and $\psi = 0$ ae on Γ_ψ. Then $\omega = [0] \in W_\Omega$.

Proof

Let $z \in \Omega$. Then

$$\omega(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\Gamma_\psi} \frac{\phi(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\Gamma_\phi} \frac{i\psi(\zeta)d\zeta}{\zeta - z}$$

due to $\phi(\zeta) = 0$ ae on Γ_ϕ and $\psi(\zeta) = 0$ ae on Γ_ψ. Therefore an equivalent function $\omega^* = \phi^* + i\psi^*$ can be formed where $\omega^* \in W_\Omega$ and

$$\phi^*(z) = \begin{cases} 0, & z \in \Gamma_\phi \\ \phi(z), & z \in \Gamma_\psi \end{cases}$$

and $\omega(z) = \omega^*(z)$ for all $z \in \Omega$.

By use of the Riemann Mapping Theorem and Caratheodory's Extension of the Riemann Mapping Theorem, any two bounded simply connected domains can be conformally mapped onto each other with a one-to-one correspondence (from the continuous extension of the conformal mapping) of the boundary points. In a recent textbook, Mathews (1982) shows that for the case of ϕ^* being constant on Γ_ϕ and $\frac{\partial \phi^*}{\partial n} = 0$ on Γ_ψ that ω^* is a constant complex number over Ω. By continuity, $\phi^* = 0$ over Ω and, by a similar argument, $\psi^* = 0$ over Ω. Thus $\omega^* = 0$ over Ω and, consequently, $\omega = 0$ over Ω. Hence, $\omega = [0]$.

7.2.5. Theorem (W_Ω is a Vector Space)

W_Ω is a linear vector space over the field of real numbers.

Proof

This follows directly from the character of analytic functions. The zero element has already been noted by [0] in Theorem 7.2.4.

7.2.6. Theorem (Definition of the Inner-Product)

Let $x,y,z \in W_\Omega$. Define a real-valued function (x,y) by

$$(x,y) = \int_{\Gamma_\phi} \text{Rex Rey } d\mu + \int_{\Gamma_\psi} \text{Imx Imy } d\mu$$

Then $(\ ,\ )$ is an inner-product over W_Ω.

Proof

It is obvious that $(x,y) = (y,x)$; $(kx,y) = k(x,y)$ for k real; $(x+y,z) = (x,z) + (y,z)$; and $(x,x) \geq 0$. By theorem 7.2.4, $(x,x) = 0$ implies $Rex = 0$ ae on Γ_ϕ and $Imx = 0$ ae on Γ_ψ and $x = [0] \in W_\Omega$.

Three theorems follow immediately from the above.

7.2.7. Theorem (W_Ω is an Inner-Product Space)

For the defined inner-product, W_Ω is an inner-product space over the field of real numbers.

7.2.8. Theorem ($||\omega||$ is a Norm on W_Ω)

A norm is defined by $||x|| = (x,x)^{\frac{1}{2}}$ for $x \in W_\Omega$.

7.2.9. Theorem

Let $x \in W_\Omega$ and $||x|| = 0$. Then $x = [0]$.

7.3. The CVEBM and W_Ω

7.3.1 Definition 7.3: (Angle Points)

Let the number of angle points of Γ be noted as Λ. By a nodal partition P_n of Γ, $m = \Lambda(n-1)$ nodes $\{z_j\}$ are defined on Γ such that a node is located at each angle-point of Γ and the remaining nodes are distributed on Γ. Nodes are numbered sequentially in a counterclockwise direction along Γ. The scale of P_n is indicated by ℓ where $\ell = \max|z_{j+1} - z_j|$. Thus n nodes are equally spaced along each line segment.

7.3.2. Definition 7.4: (Boundary Element)

A boundary element Γ_j is the line segment joining nodes z_j and z_{j+1}.

7.3.3. Theorem

Let P_n be defined on Γ. Then

$$\Gamma = \bigcup_{j=1}^{m} \Gamma_j$$

where $m = \Lambda(n-1)$

Proof

Follows from Γ being piecewise linear, and the construction of P_n.

7.3.4. Definition 7.5: (Linear Basis Function)

A linear basis function $N_j(\zeta)$ is defined for $\zeta \in \Gamma$ by

$$N_j(\zeta) = \begin{cases} (\zeta - z_{j-1})/(z_j - z_{j-1}) & , \quad \zeta \in \Gamma_{j-1} \\ (z_{j+1} - \zeta)/(z_{j+1} - z_j) & , \quad \zeta \in \Gamma_j \\ 0 & , \quad \zeta \notin \Gamma_{j-1} \cup \Gamma_j \end{cases}$$

The values of $N_j(\zeta)$ is found to be real and bounded as indicated by the next theorem.

7.3.5. Theorem

Let $N_j(\zeta)$ be defined for node $z_j \in \Gamma$. Then $0 \leq N_j(\zeta) \leq 1$.

7.3.6. Definition 7.6: (Global Trial Function)

Let P_n be defined on Γ with $m \geq \Lambda$ and with scale ℓ. At each node z_j, define nodal values $\bar{\omega}_j = \bar{\phi}_j + i\bar{\psi}_j$ where $\bar{\phi}_j$ and $\bar{\psi}_j$ are real numbers. A global trial function $G_m(\zeta)$ is defined on Γ by

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\bar{\omega}_j$$

7.3.7. Theorem

$G_m(\zeta)$ is continuous on Γ.

7.3.8. Discussion

As a result of $\omega(\zeta) \in L_2(\Gamma)$, then $\omega(\zeta)$ is measurable on Γ and for every $\varepsilon > 0$ there exists a continuous complex-valued function $g(\zeta)$ such that $||\omega(\zeta) - g(\zeta)||_1 < \varepsilon/2$. Choosing $G_m(\zeta)$ to approximate $g(\zeta)$ by $||G_m(\zeta) - g(\zeta)||_1 < \varepsilon/2$,

then $||\omega(\zeta) - G_m(\zeta)||_1 \leq ||\omega(\zeta) - g(\zeta)||_1 + ||g(\zeta) - G_m(\zeta)||_1 < \varepsilon$.

7.3.9. Theorem

Let $\omega \in W_\Omega$. For $\varepsilon > 0$ there exists a $G_m(\zeta)$ such that $||\omega(\zeta) - G_m(\zeta)||_1 < \varepsilon$.

Proof

Follows from the discussion in 7.3.8.

7.3.10. Discussion

The CVBEM approximation function $\hat{\omega}_m(z)$ is developed from the singular integral for a partition P_n of Γ by

$$\hat{\omega}_m(z) = \frac{1}{2\pi i} \int_\Gamma \frac{G_m(\zeta)d\zeta}{\zeta - z}, \quad z \in \Omega \tag{7.1}$$

where $G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\bar{\omega}_j$ is the global trial function chosen to achieve $||\omega(\zeta) - G_m(\zeta)||_1 < \varepsilon$ for $\omega \in W_\Omega$ and $\varepsilon > 0$. Expanding G_m in the integrand gives

$$\hat{\omega}_m(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_\Gamma \frac{N_j(\zeta)\bar{\omega}_j d\zeta}{\zeta - z}, \quad z \in \Omega \tag{7.2}$$

Appendix A shows that $\hat{\omega}_m(z)$ can be written as

$$\hat{\omega}_m(z) = R_1(z) + \sum_{j=1}^{m} c_j(z - z_j)\,\mathrm{Ln}(z - z_j), \quad z \in \Omega \tag{7.3}$$

where $R_1(z)$ is a first degree complex polynomial resulting from the 2π-circuit along Γ about point z; the complex logarithm is with respect to point z

(the branch cut is a ray originating from point $z \in \Omega$); and the c_j are complex constants $c_j = a_j + ib_j$ where the a_j and b_j are real numbers. The problem now can be restated as how to choose the best values for the c_j (and the $R_1(z)$ constants) such as to minimize a defined norm. Because ϕ is known only on Γ_ϕ and ψ is known only on Γ_ψ ($\omega = \phi + i\psi$), $||\omega - \hat{\omega}_m||_2$ is undefined. Therefore, the constants will be chosen to minimize the newly defined norm $||\omega - \hat{\omega}_m||$ where the goal is $||\omega - \hat{\omega}_m|| \to 0 \Rightarrow \hat{\omega}_m(z) \to \omega(z)$ for all $z \in \Omega$.

For development purposes, the $\text{Ln}(z - z_j)$ functions are replaced by $\text{Ln}_j(z - z_j)$ functions where logarithm branch cuts are rays from each z_j which lie exterior of $\bar{\Omega} - \{z_j\}$ (see Appendix A).

Letting $R_1(z) = c_{m+1} + c_{m+2}z$, the CVBEM approximation used is now defined as

$$\hat{\omega}_m(z) = \sum_{j=1}^{m+2} c_j T_j \tag{7.4}$$

where

$$T_j = \begin{cases} (z - z_j)\,\text{Ln}_j\,(z - z_j)\,, & j = 1,2,\cdots,m \\ 1 + 0i\,, & j = m+1 \\ z\,, & j = m+2 \end{cases}$$

By the use of the $\text{Ln}_j(z - z_j)$ functions, $\hat{\omega}_m(z)$ is analytic over $\bar{\Omega}$ except at the nodal points, and $\hat{\omega}_m(z)$ is continuous over $\bar{\Omega}$. In fact, $\hat{\omega}_m(z)$ is analytic everywhere except along the branch cuts.

If $c_j = a_j + ib_j$ is substituted into (7.4), the CVBEM approximation can be written with respect to real number coefficients γ_j as

$$\hat{\omega}_m(z) = \sum_{j=1}^{2(m+2)} \gamma_j f_j \tag{7.5}$$

where the f_j functions are given by

$$\left.\begin{aligned} f_1 &= (z - z_1)\,\mathrm{Ln}_1\,(z - z_1) \\ f_2 &= if_1 \\ &\vdots \\ f_{2m-1} &= (z - z_m)\,\mathrm{Ln}_m\,(z - z_m) \\ f_{2m} &= if_{2m-1} \\ f_{2m+1} &= 1 \\ f_{2m+2} &= i \\ f_{2m+3} &= z \\ f_{2(m+2)} &= iz \end{aligned}\right\} \qquad (7.6)$$

7.3.11. Theorem (Linear Independence of Nodal Expansion Functions)

The set of functions $\{(z - z_j)\,\mathrm{Ln}_j\,(z - z_j),\ j = 1,2,\cdots,m+1\}$ are linearly independent.

Proof

Suppose the first m functions are linearly independent, but the (m+1)-function is linearly dependent on the other m functions. Then for complex constants c_j,

$$c_{m+1}(z - z_{m+1})\,\mathrm{Ln}_{m+1}(z - z_{m+1}) = \sum_{j=1}^{m} c_j(z - z_j)\,\mathrm{Ln}_j(z - z_j)$$

Taking the second derivative with respect to z gives

$$\frac{c_{m+1}}{(z - z_{m+1})} = \sum_{j=1}^{m} \frac{c_j}{(z - z_j)}, \quad \text{for } z \neq z_k,\ k = 1,2,\cdots,m+1$$

Rearranging terms, the above implies that

$$c_{m+1} \prod_{k=1}^{m} (z - z_k) = (z - z_{m+1}) \sum_{j=1}^{m} c_j \prod_{\substack{k=1 \\ k \neq j}}^{m} (z - z_k)$$

which is valid only if $c_k = 0$ for each $k = 1,2,\cdots,m+1$.

7.3.12. Discussion

From the previous theorem, the set of functions $\{T_j\}$ of (7.4) are also linearly independent and, more importantly in this development, the $\{f_j\}$ are linearly independent with respect to the real number field. Thus for a given number m of nodes on Γ, the functions $\{f_j;\ j = 1,2,\cdots,m\}$ forms a basis for the vector space spanned by the $\{f_j\}$, noted by $\hat{W}_\Omega^m$. In this notation, m indicates the number of nodes defined on Γ (always, $m \geq \Lambda$), and the hat indicates the CVBEM approximation function vector space.

The CVBEM objective is to choose a $\hat{\omega}_m \in \hat{W}_\Omega^m$ which minimizes $||\omega - \hat{\omega}_m||$ where $\omega \in W_\Omega$ and the nodes $\{z_j\}$ are fixed on Γ.

7.3.13. Theorem

Let $\omega \in W_\Omega$ and $z \in \Omega$. For every $\varepsilon > 0$ there exists a CVBEM approximation $\hat{\omega}_m$ such that $|\omega(z) - \hat{\omega}_m(z)| < \varepsilon$.

Proof

Let $d = \min |\zeta - z|$, $\zeta \in \Gamma$. Then for a global trial function $G_m(\zeta)$ defined on Γ

$$|\omega(z) - \hat{\omega}_m(z)| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{[\omega(\zeta) - G_m(\zeta)]d\zeta}{\zeta - z} \right|$$

$$\leq \frac{1}{2\pi d} ||\omega - G_m||_1 \leq \frac{\sqrt{L}}{2\pi d} ||\omega - G_m||_2$$

Choosing G_m (see section 7.3.10) such that $||\omega - G_m||_2 < 2\pi d\ \varepsilon/\sqrt{L}$ (or $||\omega - G_m||_1 < 2\pi d\ \varepsilon$) guarantees the desired result.

More insight as to the power of the CVBEM is provided by an analogy to convergence in measure:

7.3.14. Theorem

Let $\varepsilon > 0$. Then there exists a $0 < \delta < 1$ such that the

$$\iint_{\Omega-\Omega_\delta} d\Omega < \varepsilon \quad \text{and} \quad \lim_{\substack{\ell\to 0 \\ m\to\infty}} |\omega(z) - \hat{\omega}_m(z)| = 0.$$

Proof

Choose $0 < \delta < 1$ such that the area of $\Omega - \Omega_\delta$ is less than ε. Let $d = (1-\delta)\min|\zeta|$, $\zeta \in \Gamma$ where $\omega \in W_\Omega$. Then by Theorem 7.3.13, the required result follows.

7.3.15. Discussion

The above theorems discuss the existence of a CVBEM approximation $\hat{\omega}_m(z)$ which converges in measure to $\omega(z)$. That is, for an arbitrarily small (1-δ)-strip inside of Γ, $\hat{\omega}_m(z) \to \omega(z)$ for all $z \in \bar{\Omega}_\delta$ as $m\to\infty$ and $\ell\to 0$. To develop the CVBEM approximation $\hat{\omega}_m(z)$, the defined norm $||x||$ for $x \in W_\Omega$ is used.

To proceed, the $\{f_j\}$ are orthonormalized by the Gram-Schmidt procedure to the set of functions $\{g_j\}$ using the defined inner-product on W_Ω. That is, $g_1 = f_1/||f_1||$, $g_2 = (f_2 - (f_2, g_1)g_1)/||f_2 - (f_2, g_1)g_1||$, and so forth. With respect to $\{g_j\}$,

$$\hat{\omega}_m(z) = \sum_{j=1}^{2(m+2)} \hat{\gamma}_j g_j(z)$$

where the $\hat{\gamma}_j$ are generalized Fourier coefficients to be determined. It is noted that the $g_k(z)$ are finite combinations of the f_j-functions.

The value of $||\omega - \hat{\omega}_m||$ is minimized when $\hat{\gamma}_j = (\omega, g_j)$. By back-substitution, the γ_j corresponding to the $\{f_j\}$ can be evaluated. In this fashion, the CVBEM approximator $\hat{\omega}_m(z)$ is developed for $\omega \in W_\Omega$ and the provided boundary conditions of ϕ defined on Γ_ϕ and ψ defined on Γ_ψ.

Because W_Ω is an inner-product space with the defined inner-product, Bessel's inequality applies.

7.4. The Space W_Ω^A

7.4.1. Definition 7.7: (W_Ω^A)

A subspace of W_Ω are those elements which are analytic over $\bar{\Omega}$. Thus, $\omega \in W_\Omega^A$ implies ω is analytic over $\Omega \cup \Gamma$.

7.4.2. Theorem

W_Ω^A is a linear vector space over the field of real numbers.

Proof

Follows from the parent space W_Ω. However, it is noted that ae equality is unnecessary due to $\omega \in W_\Omega^A$ implies continuity over $\bar{\Omega}$.

7.4.3. Theorem

W_Ω^A is an inner-product space using the defined inner-product.

Proof

Of interest is showing $(x,x) = 0 \Rightarrow x = 0$. Green's theorem gives

$$\int_\Omega (\phi_x^2 + \phi_y^2)\, d\Omega = \int_\Gamma \phi \frac{\partial \phi}{\partial n}\, d\Gamma + \int_\Omega \phi \nabla^2 \phi\, d\Omega$$

where ϕ_x and ϕ_y are partial derivatives of $\phi(x,y)$ in the x- and y-direction, and $\frac{\partial \phi}{\partial n}$ is a normal derivative along Γ. But $\left|\frac{\partial \phi}{\partial n}\right| = \left|\frac{\partial \psi}{\partial s}\right|$ where s is a tangential coordinate along Γ and the Cauchy-Riemann relations apply.

Thus $\nabla^2\phi = 0$ over Ω due to $\omega = \phi + i\psi$ and $\omega \in W_\Omega^A$. Also, $\phi = 0$ on Γ_ϕ and $\frac{\partial\psi}{\partial s} = 0$ on Γ_ψ by assumption.

Thus $\int_\Omega (\phi_x^2 + \phi_y^2)d\Omega = 0$ and $\phi(x,y)$ is constant over Ω. By continuity, $\phi = 0$ over $\bar{\Omega}$. Similarly $\psi = 0$ over $\bar{\Omega}$, and $\omega = 0$.

7.4.4. Discussion

For $\omega \in W_\Omega^A$ and $z \in \Omega$, Cauchy's theorem gives immediately that

$$\omega(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)d\zeta}{\zeta - z}, \quad z \in \Omega \tag{7.7}$$

Letting $G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\,\omega_j$ where $\omega_j = \omega(z_j)$, then $\lim_{\substack{m\to\infty \\ \ell\to 0}} G_m(\zeta) = \omega(\zeta)$

and

$$\omega(z) = \lim_{\substack{m\to\infty \\ \ell\to 0}} \frac{1}{2\pi i}\int_\Gamma \frac{G_m(\zeta)d\zeta}{\zeta - z}, \quad z \in \Omega \tag{7.8}$$

(A detailed proof of this convergence is in Appendix B.)

Thus for $z \in \Omega$,

$$\omega(z) = c_o + c_{-1}\, z + \sum_{j=1}^{\infty} c_j\,(z - z_j)\,\mathrm{Ln}_j\,(z - z_j), \quad z \in \Omega \tag{7.9}$$

where now c_o and c_{-1} are also complex constants. It can also be argued that the $c_o + c_{-1}\, z$ terms can be eliminated entirely when using the infinite series expansion.

Because $\omega(z) = \lim_{\substack{m\to\infty \\ \ell\to 0}} \hat{\omega}_m(z)$ over Ω, then the boundary values of the limiting CVBEM approximator (taking in the limit as $\delta\zeta\to\zeta$ for each $\zeta \in \Gamma$) equal the boundary values of $\omega \in W_\Omega^A$.

Writing the $\omega(z)$ function with respect to the Gramm-Schmidt orthonormalized functions $\{g_j\}$ of Section 7.3 (with respect to the defined inner-product)

$$\omega(z) = \sum_{j=1}^{\infty} (\omega, g_j) g_j(z), \quad z \in \Omega. \tag{7.10}$$

7.4.5. Theorem

The set $\{g_j\}$ is complete.

Proof

Suppose $\omega \in W_\Omega^A$ and $(\omega, g_j) = 0$ for every j. Then from (7.10)

$$\omega(z) = \sum_{j=1}^{\infty} (\omega, g_j) g_j = 0, \quad z \in \Omega$$

Thus $\omega(z)$ is the zero element of W_Ω^A in that in the limit as $\delta\zeta\to\zeta$, $\phi = 0$ on Γ_ϕ and $\psi = 0$ on Γ_ψ where $\omega = \phi + i\psi$. Thus the set $\{g_j\}$ is complete.

7.4.6. Theorem

Let $\omega \in W_\Omega^A$. Then ω satisfies the Dirichlet conditions for generalized Fourier series.

Proof

By assumption, there are a finite number of line segments composing Γ_ϕ and Γ_ψ. Because ω is analytic on Γ, then the boundary condition functions $B(\zeta)$ and $B'(\zeta)$ are both piecewise continuous on Γ.

7.4.7. Discussion: Another Look at W_Ω

By Theorem 7.4.6, the CVBEM will converge to the boundary values where continuous, and to the midpoint value of the discontinuity where discontinuous. Because $\hat{\omega}_m(z)$ is analytic over Ω as $m\to\infty$ (Appendix B), then also $\hat{\omega}_m(z)\to\omega(z)$ as $m\to\infty$. But by Definition 7.2.1, $\omega(\delta\zeta)\to\omega(\zeta)$ in $L_2(\Gamma)$. Due to $\omega(\delta\zeta)$ being analytic over Ω, we immediately have $\hat{\omega}_m(z)$ approximates $\omega(\delta\zeta)$ which, in turn, approximates $\omega(z)$ arbitrarily close in $L_2(\Gamma)$.

7.5. Applications

7.5.1. Introduction

A FORTRAN computer program was prepared based on the least-square boundary fit described in the previous sections. Matrix solution routines are not needed due to the orthonormal vector technique. The program was prepared to accomodate analytic function equivalents for sources, sinks, flux boundary conditions (i.e. tangential derivatives of the stream function ψ), and dissimilar regions. The program listing is contained in section 7.6.

7.5.2. Nodal Point Placement on Γ

The program operates upon an initial nodal point placement to develop the CVBEM approximation. Then, the user enters (by the CRT) x,y-coordinates for the next node location on Γ and the program computes Bessel's inequality. By randomly spotting the nodal location on Γ, Bessel's inequality is subsequently minimized and the optimum choice for the next node on Γ is made. In this fashion $\hat{\omega}_m(z)\to\omega(z)$ as $m\to\infty$.

7.5.3. Flow-Field (Flow-Net) Development

By entering x,y-coordinates, $\hat{\omega}_m(z)$ values are computed and the flow-net can be plotted with respect to the approximation $\hat{\phi}_m(z)$ and $\hat{\psi}_m(z)$ values.

Such flow-nets are included in the provided applications.

7.5.4. Approximate Boundary Development

Hromadka (1984) details the "approximate boundary" Γ technique for CVBEM error evaluation. The contour $\hat{\Gamma}$ represents the location where $\hat{\omega}_m(z)$ achieves the boundary conditions of $\omega(z)$ on Γ. That is, if the provided boundary conditions are level curves of $\omega(z)$ on Γ, then $\hat{\Gamma}$ represents the corresponding level curves of $\hat{\omega}_m(z)$. Hence if the approximate boundary $\hat{\Gamma}$ lies "sufficiently close" to Γ, the analyst can conclude that an adequate approximation has been developed. This error evaluation technique is very useful due to the ease of interpretation. Even beginners can develop highly accurate CVBEM approximations by simply observing the relationship of $\hat{\Gamma}$ to Γ, and adding nodes to Γ where departures are considered unacceptable. In the included example problems, approximate boundaries are developed for each test problem.

7.5.5. Applications

EXAMPLE 7.1: (Ideal Fluid Flow Around a Cylindrical Corner)

Ideal fluid flow around a cylindrical corner has the analytic solution of $\omega(z) = z^2 + z^{-2}$. Figure 7.1(a) depicts the problem geometry and specified boundary conditions. Figures 7.1(b) and 7.1(c) show the error plots in matching boundary values for both the known and unknown boundary conditions. Figure 7.2 shows the CVBEM computed flow net.

EXAMPLE 7.2: (Irregular Domain)

Figure 7.3 shows an irregular two-dimensional cross-section with boundary conditions. The purpose of this example is to show how the approximate boundary is used to evaluate computational error for an irregular section problem. Figure 7.4 shows a very good match between the exact and approximate boundaries.

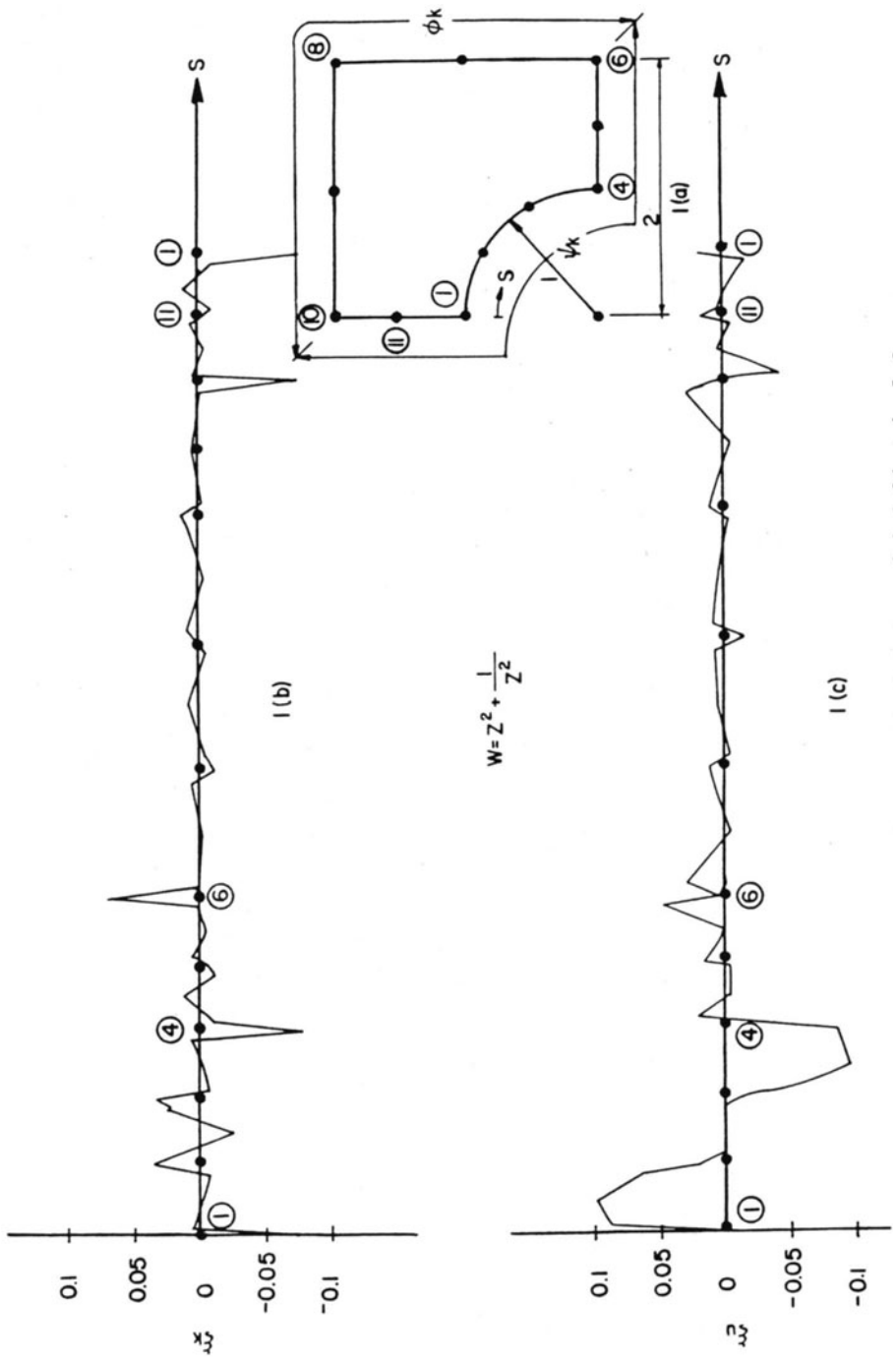
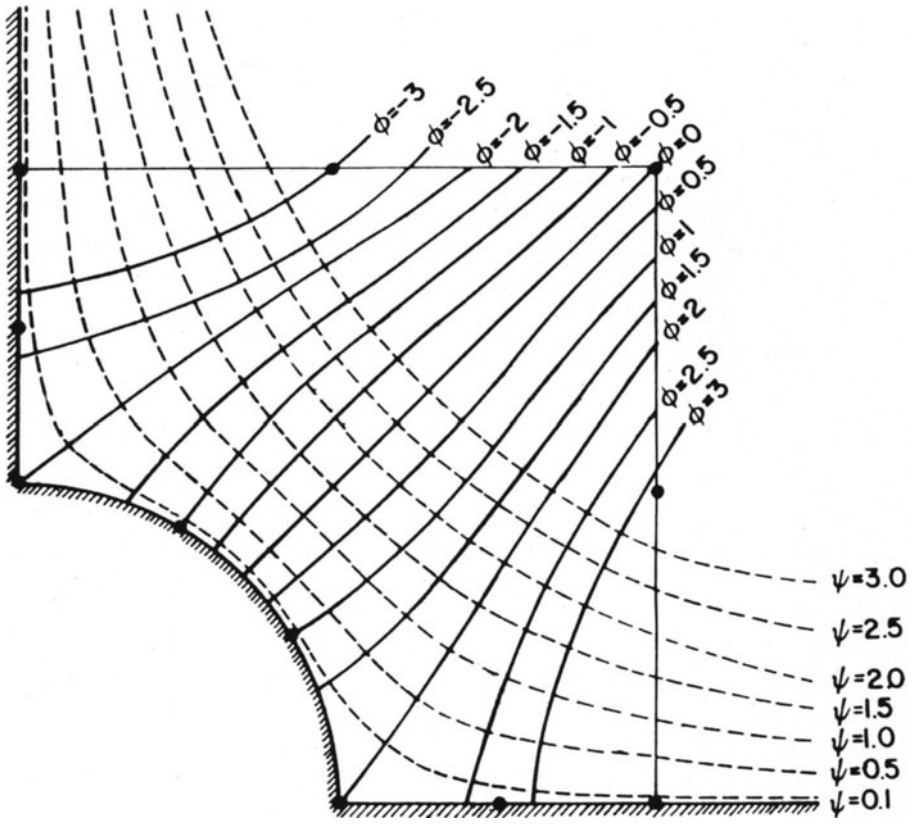


Fig. 7.1. Error Plots for Ideal Fluid Flow Around A Cylindrical Corner



$$W = Z^2 + \frac{1}{Z^2}$$

Fig. 7.2. Computed Flow Net for Ideal Fluid Flow Around a Cylindrical Corner

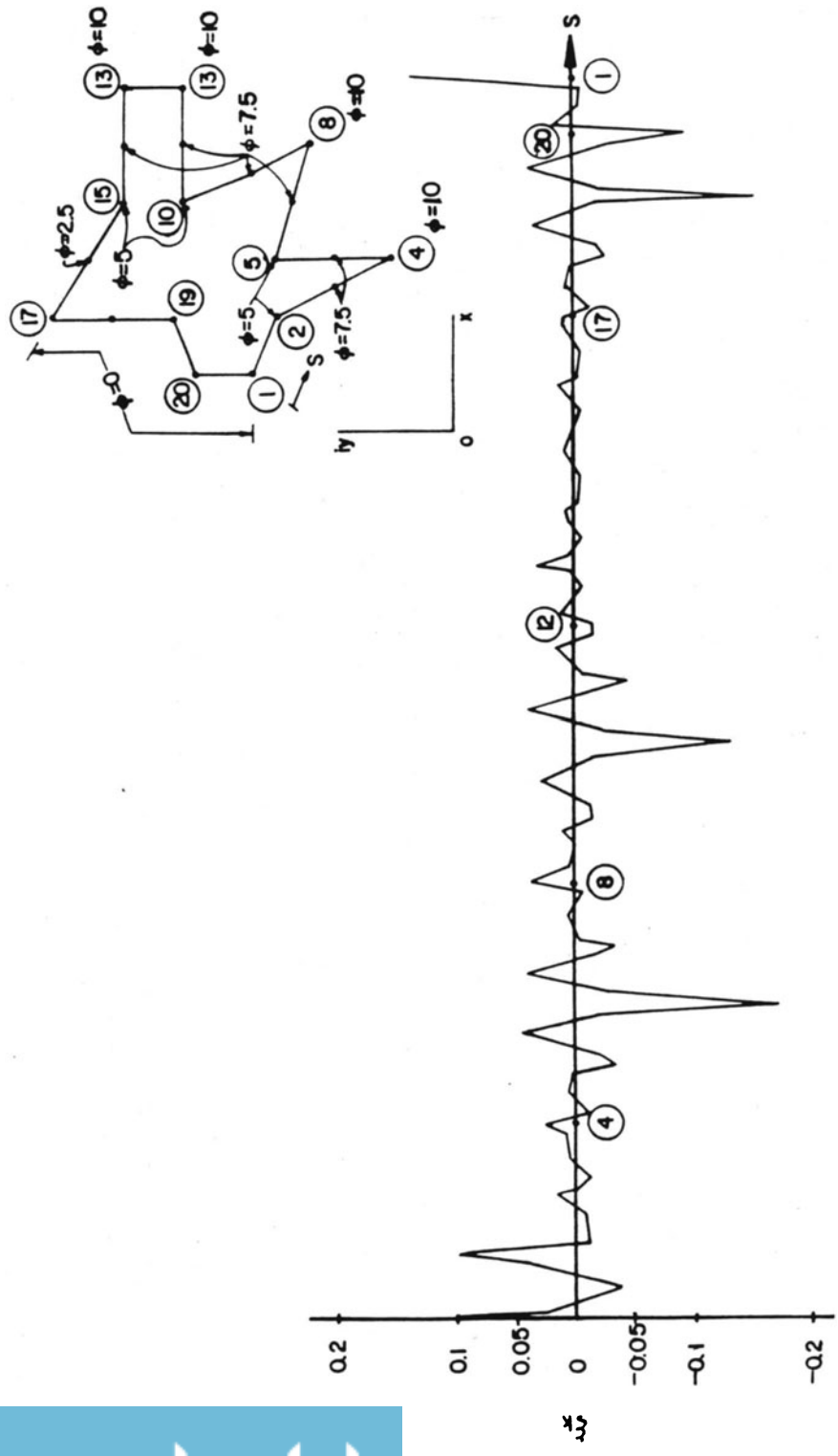


Fig. 7.3. Error Plot for Irregular Two-Dimensional Cross-section

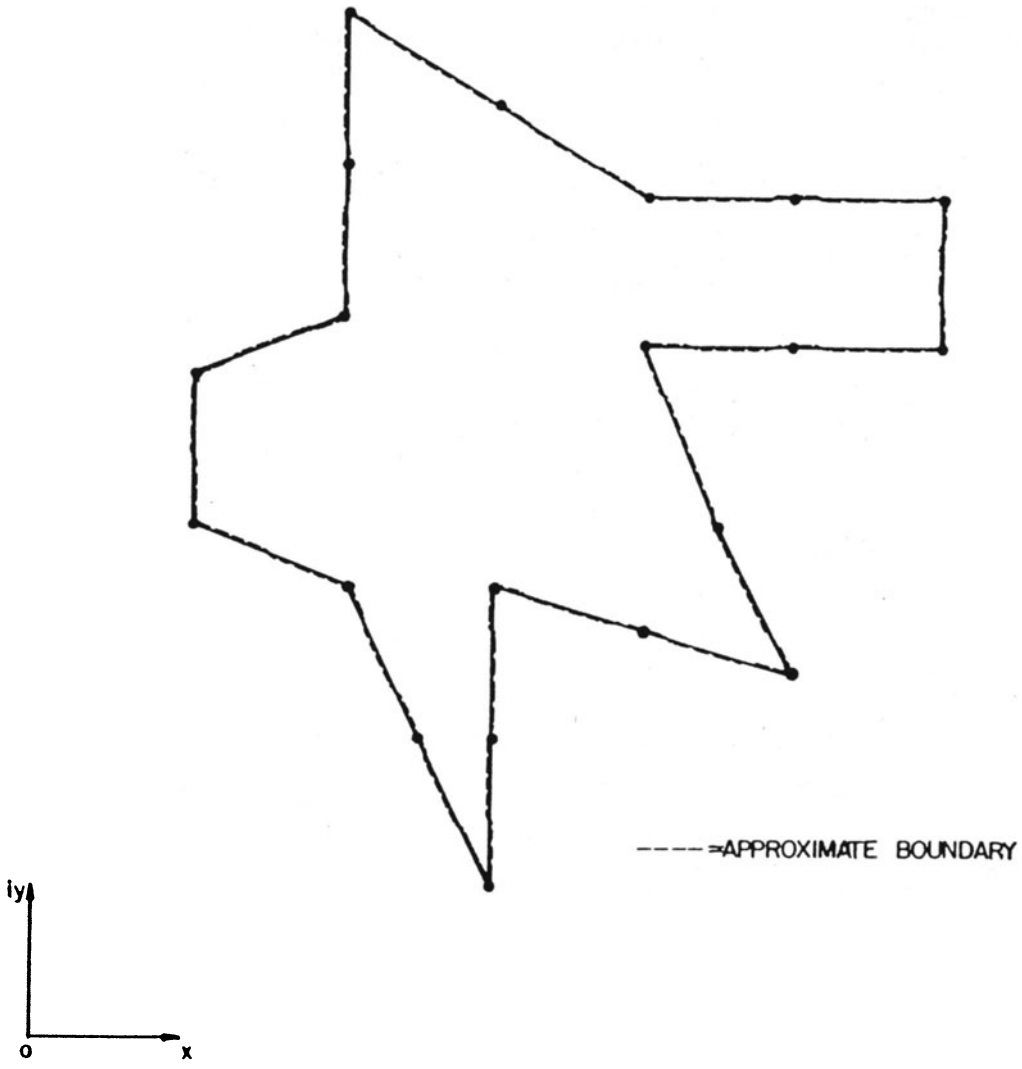


Fig. 7.4. Approximate Boundary for Irregular Two-Dimensional Cross-section

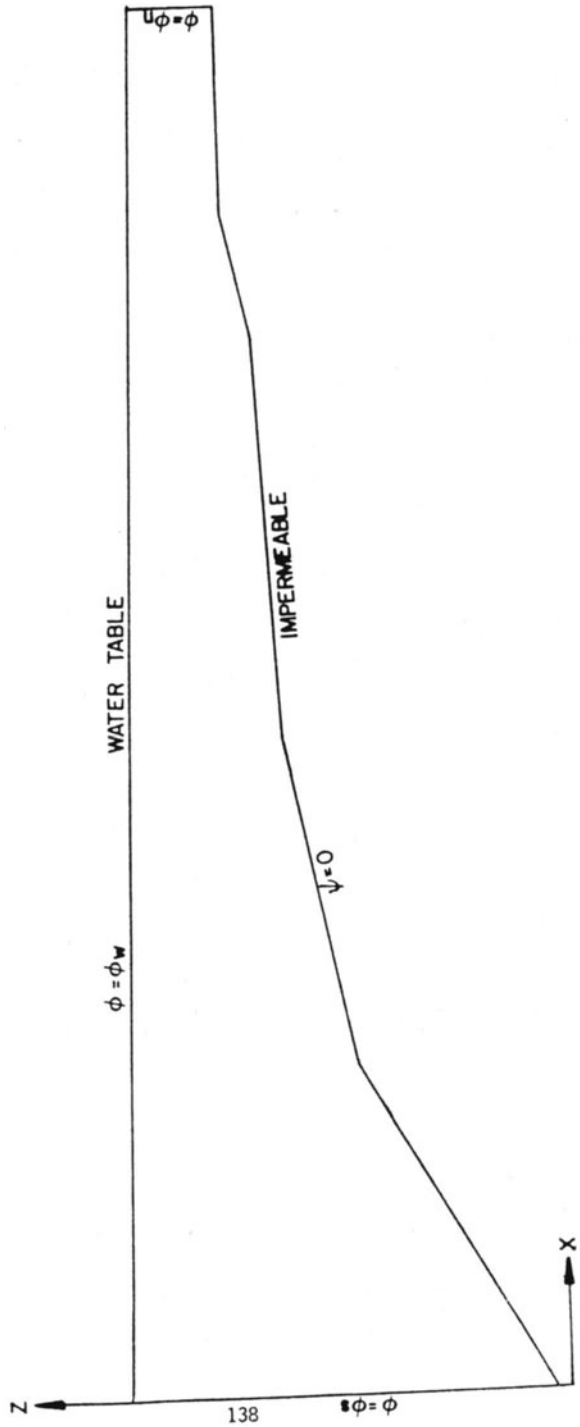


Fig. 7.5. Boundary Conditions for Shallow Unconfined Aquifer

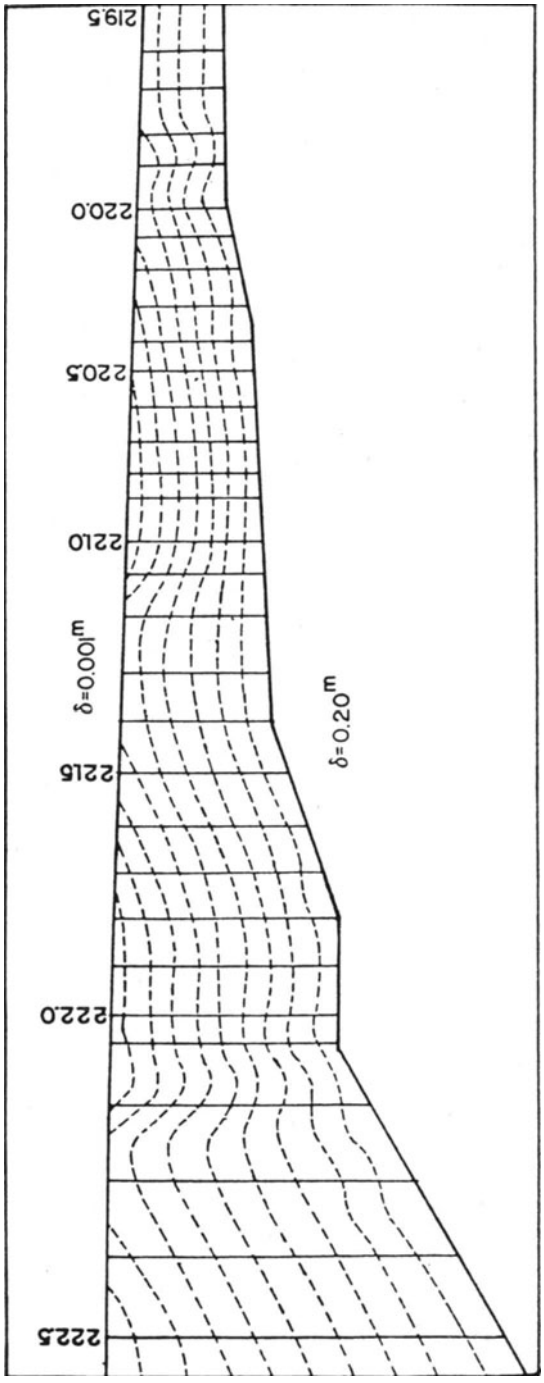


Fig. 7.6. Computed Flow Net for Shallow Unconfined Aquifer

EXAMPLE 7.3: (Long Shallow Aquifer Groundwater Problem)

A long and shallow unconfined aquifer (see Figure 7.5) is used to compare the results between the dual formulation technique (Frind et al., 1985) and the proposed CVBEM technique. The mean deviation between the exact and approximate boundary is about 0.001^m and 0.2^m for the water table and impervious boundary, respectively. The 0.2^m deviation is based on the 10^{-4} magnitude difference between the exact and approximate boundary. If this magnitude increases to 10^{-2}, the approximate boundary can be considered to coincide with the exact boundary. The equipotential lines shown on Figure 7.6 approximates those shown on Figures 6 and 8 in Frind et al.'s (1985) paper. The stream lines are not orthogonal to the equipotential lines because of the difference scales in x- and y-directions.

7.6. Computer Program: Two-Dimensional Potential Problems Using Analytic Basis Functions (CVBEM)

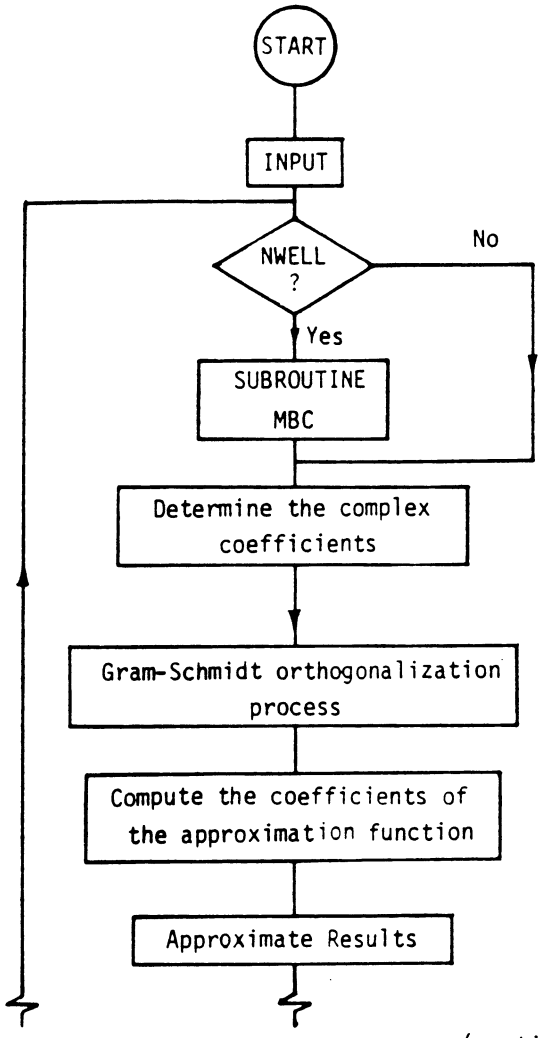
7.6.1. Introduction

The CVBEM program consists of two programs--CVBEM1 and CVBEM2. In CVBEM1, boundary conditions are approximated in a "mean-square" error sense and then minimized by the selection of complex coefficients to be associated to each nodal point located on the problem boundary, Γ.

The approximate boundary and flow net analysis can be conducted by using CVBEM2.

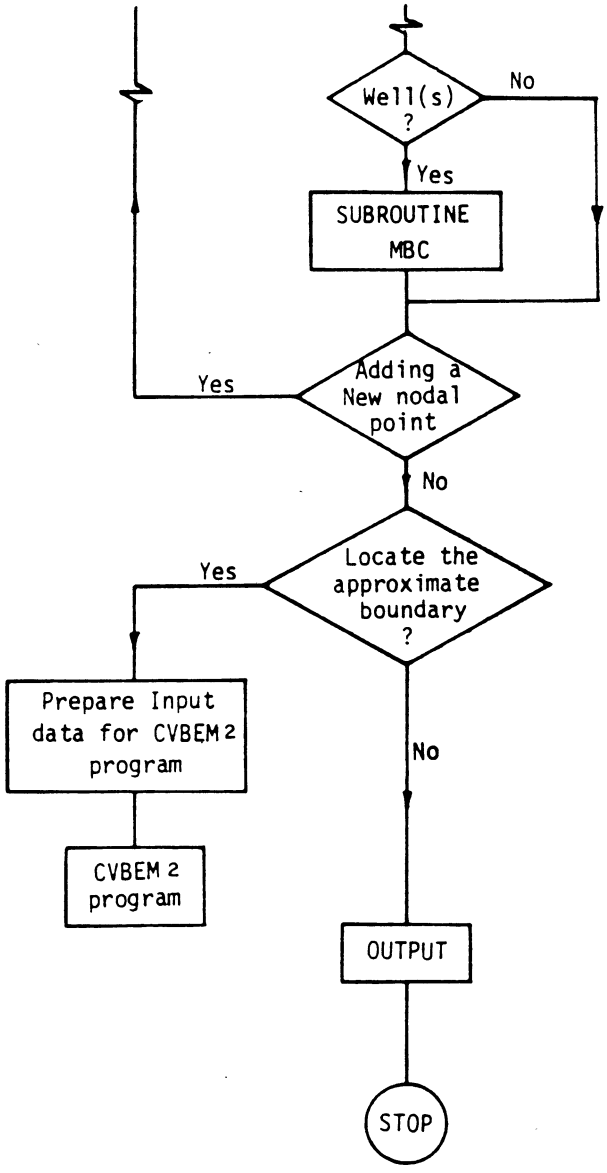
7.6.2. CVBEM1 Program Listing

Figure 7.7 depicts the simple flow chart for CVBEM1 program.



(continued next page)

Fig. 7.7. Simple Flow Chart for Program CVBEM1



```
C
C  MAIN PROGRAM
C
C  THIS IS A GENERALIZE FOURIER SERIES ANALYSIS
C  WHICH USES CVBEM H1 APPROXIMATE FUNCTION TO SOLVE THE
C  LAPLACE EQUATION
C
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      COMMON/BLK 1/ X(80),Y(80)
      COMMON/BLK 2/ ANGLE(80),KTYPE(80)
      COMMON/BLK 3/ VALUE(80,2)
      COMMON/BLK 4/ WB(60),B(60)
      COMMON/BLK 5/ G(45,60)
      COMMON/BLK 6/ XK(45,45)
      COMMON/BLK 7/ QX(10),QY(10),Q(10)
C
C  OPEN DATA FILES
C
      NRD=1
      NWT=2
      NWD=3
      NTMR=11
      NTMW=10
      KOK=0
      KADD=0
600   OPEN NRD,"LTWO.DAT"
      IF(KADD.EQ.0) OPEN NWT,"LTWO.ANS"
C
C READ INPUT DATA
C
      PI=3.141592653
      READ FREE(NRD) NNOD,NTEST,KODE,KLIN
      IF(KADD.NE.1) GO TO 610
611   WRITE(NTMW,32)
      READ FREE(NTMR) NODE
      IF(NODE.GT.NNOD)WRITE(NTMW,929)
      IF(NODE.GT.NNOD)GO TO 611
      DO 620 I=1,NODE
620   READ FREE(NRD) X(I),Y(I),KTYPE(I),VALUE(I,1),VALUE(I,2)
     1               ,ANGLE(I)
      K=NODE+1
      IF(K.LE.NNOD)GO TO 621
      X(K)=.5*(X(NODE)+X(1))
      Y(K)=.5*(Y(NODE)+Y(1))
      KTYPE(K)=MIN0(KTYPE(NODE),KTYPE(1))
      ANGLE(K)=ANGLE(NODE)
      DD=DSQRT((X(K)-X(NODE))**2+(Y(K)-Y(NODE))**2)
      DDD=DSQRT((X(1)-X(NODE))**2+(Y(1)-Y(NODE))**2)
      DO 647 I=1,2
647   VALUE(K,I)=VALUE(NODE,I)+DD*(VALUE(1,I)-VALUE(NODE,I))/DDD
      KK=K+1
      GO TO 623
621   KK=K+1
      READ FREE(NRD) X(KK),Y(KK),KTYPE(KK),VALUE(KK,1),VALUE(KK,2)
     1               ,ANGLE(KK)
      X(K)=.5*(X(NODE)+X(KK))
      Y(K)=.5*(Y(NODE)+Y(KK))
      KTYPE(K)=MIN0(KTYPE(NODE),KTYPE(KK))
      ANGLE(K)=ANGLE(NODE)
      DD=DSQRT((X(K)-X(NODE))**2+(Y(K)-Y(NODE))**2)
      DDD=DSQRT((X(KK)-X(NODE))**2+(Y(KK)-Y(NODE))**2)
      DO 640 I=1,2
640   VALUE(K,I)=VALUE(NODE,1)+DD*(VALUE(KK,I)-VALUE(NODE,I))/DDD
      KK=KK+1
623   NNOD=NNOD+1
      IF(KK.GT.NNOD)GO TO 643
      GO TO 641
643   READ FREE(NRD) NWELL
      IF(NWELL.EQ.0)GO TO 650
```

```
      DO 645 I=1,NWELL
645   READ FREE(NRD) QX(I),QY(I),Q(I)
      GO TO 650
641   DO 630 I=KK,NNOD
630   READ FREE(NRD) X(I),Y(I),KTYPE(I),VALUE(I,1),VALUE(I,2)
     1               ,ANGLE(I)
      READ FREE(NRD) NWELL
      IF(NWELL.EQ.0) GO TO 650
      DO 631 I=1,NWELL
631   READ FREE(NRD) QX(I),QY(I),Q(I)
      GO TO 650
610   DO 7 I=1,NNOD
      READ FREE(NRD) X(I),Y(I),KTYPE(I),VALUE(I,1),VALUE(I,2)
     1               ,ANGLE(I)
7     CONTINUE
      READ FREE(NRD) NWELL
      IF(NWELL.EQ.0) GO TO 615
      DO 613 I=1,NWELL
613   READ FREE(NRD) QX(I),QY(I),Q(I)
615   WRITE(NWT,2)
      DO 9 I=1,NNOD
      IF(KTYPE(I).EQ.1)VALUE(I,2)=0.
      IF(KTYPE(I).EQ.2)VALUE(I,1)=0.
      WRITE(NWT,8) I,X(I),Y(I),KTYPE(I),VALUE(I,1),VALUE(I,2)
     1             ,ANGLE(I)
9     CONTINUE
      IF(NWELL.EQ.0) GO TO 619
      WRITE(NWT,82)
      DO 85 I=1,NWELL
85    WRITE(NWT,83) I,QX(I),QY(I),Q(I)
619   IF(KOK.EQ.1)GO TO 617
C
C  OUTPUT FORMATS
C
2     FORMAT(//,10X,'*** BOUNDARY NODE DATA ***',
     1//,1X,'NODE',3X,'X(I)',6X,'Y(I)',4X,'KTYPE(I)',5X,
     2'VAUE(I)',9X,'OUTER',/,2X,'NO.',21X,'1=SV;2=SF',3X,
     3'SV',5X,'SF',8X,'NORMAL',/,27X,'3=SV&SF',20X,'ANGLE(I)')
3     FORMAT(2X,2I3,3X,D10.4)
4     FORMAT(/.10X,'*** APPROXIMATE NODAL VALUES AND ERRORS ***',
     1//,6X,'NODE',9X,'STATE',12X,'STREAM',/,
     25X,'NUMBER',6X,'VARIABLE',10X,'FUNCTION',12X,'ERROR',/)
45    FORMAT(/,10X,'*** APPROXIMATE POINT VALUES AND ERRORS --',
     1' FUNCTION FORM ***',//,5X,'POINT',9X,'STATE',12X,'STREAM',/,
     25X,'NUMBER',6X,'VARIABLE',10X,'FUNCTION',12X,'ERROR',/)
5     FORMAT(3X,I5,3(8X,D10.4))
8     FORMAT(1X,I3,2X,F8.3,2X,F8.3,5X,I2,4X,F7.2,1X,F7.2,5X,F6.2)
21    FORMAT(//,10X,'*** EVALUATION POINT DATA ***',
     1//,1X,'POINT',2X,'X(I)',6X,'Y(I)',4X,'KTYPE(I)',5X,
     2'VALUE(I)',/,2X,'NO.',21X,'1=SV;2=SF',3X,'SV',5X,'SF',/,
     327X,'3=SV&SF')
22    FORMAT(10(1X,D10.4))
23    FORMAT(/,10X,'*** NODE NUMBER #',I3,2X,'(REAL,IMAGINARY)')
24    FORMAT(/,10X,'*** NODAL POINT VECTOR EXPANSION, F(I) ***',/)
25    FORMAT(/,10X,'*** ORTHOGONAL VECTOR EXPANSION, G(I) ***',/)
26    FORMAT(/,10X,'*** ORTHOGONAL TEST (G(I),G(I)) ***',//,
     1'    I  J  (G(I),G(J))')
28    FORMAT(/,10X,'*** EVALUATION COEFFICIENTS,',
     1' (G(I),B(I))/(G(I),G(I)) ***',/)
29    FORMAT(/,10X,'*** BACKSUBSTITUTION COEFFICIENTS ***',/)
31    FORMAT(/,2X,'ENTER A [1] FOR ADDING AN ADDITIONAL NODAL POINT')
32    FORMAT(/,2X,'ENTER THE NODE NUMBER THAT',/,
     12X,'THE ADDITIONAL NODAL POINT WILL FOLLOWED')
33    FORMAT(/,2X,'ENTER THE X- AND Y- COORDIATES',/,
     12X,'FOR THE ADDITIONAL NODAL POINT')
34    FORMAT(/,2X,'THE ADDITIONAL NODAL POINT (',F8.4,',',F8.4,
     1') ',/,'  HAS THE NORM EQUAL TO ',D10.4)
35    FORMAT(/,2X,'ENTER A [1] TO ACCEPT THIS ADDITIONAL NODAL POINT')
```

```
 36    FORMAT(/,2X,'*** BESSEL INEQUALITY ***',/,2X,D10.4,
      1' >= ',D10.4,' AND THE DIFFERENCE IS ',D10.4,/,120('-'),/)
 38    FORMAT(2(1X,F10.4),1X,I2,3(1X,F10.4))
 71    FORMAT(2X,'ENTER A [1] TO EVALUATE THE STATE VARIABLE AND',
      1/,'  STREAM FUNCTION FOR A GIVEN POINT')
 72    FORMAT(2X,'===> EXECUTE PROGRAM "CCYLTWO1" TO EVALUATE',/,
      1'  STATE VARIABLE AND STREAM FUNCTION FOR A GIVEN POINT')
 81    FORMAT(1X,I3,2X,F8.3,2X,F8.3,5X,I2,4X,F7.2,1X,F7.2)
 82    FORMAT(/,2X,'*** LOCATION AND STRENGTH OF WELL(S) ***',/)
 83    FORMAT(1X,I3,2(2X,F8.3),7X,F10.5)
 99    FORMAT(/,120('='))
919    FORMAT(2X,'DATA POINT HAS BEEN USED...TRY ANOTHER POINT')
929    FORMAT(2X,'NUMBER EXCEEDS TOTAL NODE NUMBER...TRY AGAIN')
C
C  CALCULATE X-COORINATE, Y-COORIDNATE, AND BOUNDARY VALUES
C  FOR EVALUATION POINTS
C
650    K=NNOD
307    DO 300 I=1,NNOD
       IP1=I+1
       IF(IP1.GT.NNOD)IP1=1
       X1=X(I)
       Y1=Y(I)
       X2=X(IP1)
       Y2=Y(IP1)
       DX=X2-X1
       DY=Y2-Y1
       R11=VALUE(I,1)
       R12=VALUE(I,2)
C..MODIFY THE EFFECT BY THE WELL(S)
       IF(NWELL.NE.0) CALL MBC(NWELL,R11,R12,X1,Y1,1)
       R21=VALUE(IP1,1)
       R22=VALUE(IP1,2)
C..MODIFY THE EFFECT BY THE WELL(S)
       IF(NWELL.NE.0) CALL MBC(NWELL,R21,R22,X2,Y2,1)
       DR1=R21-R11
       DR2=R22-R12
       DO 300 J=1,NTEST
       K=K+1
       P=1./(2.*(NTEST+1))
       IF(NTEST.EQ.1) RATIO=0.5
       IF(NTEST.GT.1) RATIO=P+(1.-2.*P)*FLOAT(J-1)/(FLOAT(NTEST)-1.)
       X(K)=X1+DX*RATIO
       Y(K)=Y1+DY*RATIO
       ANGLE(K)=ANGLE(I)
       VALUE(K,1)=R11+DR1*RATIO
       VALUE(K,2)=R12+DR2*RATIO
       KTYPE(K)=MIN0(KTYPE(I),KTYPE(IP1))
300    CONTINUE
       NTOT=NNOD*NTEST
       NTOTL=2*NNOD
       NNODP=NNOD
       IF(KLIN.EQ.1)NTOTL=2*NNOD+4
       IF(KLIN.EQ.1)NNODP=NNOD+2
617    IF(KADD.EQ.0 .OR. KOK.EQ.1)WRITE(NWT,21)
       SAREA=0.
       DO 310 KK=1,NTOT
       I=KK+NNOD
       IF(KTYPE(I).EQ.1)VALUE(I,2)=0.
       IF(KTYPE(I).EQ.2)VALUE(I,1)=0.
       SAREA=SAREA+VALUE(I,1)**2+VALUE(I,2)**2
       IF(KADD.EQ.0 .OR. KOK.EQ.1)WRITE(NWT,81)KK,X(I),Y(I),KTYPE(I),
      1                           VALUE(I,1),VALUE(I,2)
       IF(KOK.EQ.1)GO TO 310
       ANGLE(I)=ANGLE(I)*PI/180.
310    CONTINUE
```

```
C
C  DETERMINE THE COEFFICIENTS OF ALPHA'S AND BETA'S FOR
C  COMPLEX VARIABLE (Z-Z(J))*LN(Z-Z(J)) ARITHMATIC
C
      IF(KOK.EQ.1)GO TO 660
      IF(KODE.EQ.1)WRITE(NWT,99)
      IF(KODE.EQ.1)WRITE(NWT,24)
      DO 320 I=1,NNODP
      II=2*I
      IIM1=II-1
      XX=X(I)
      YY=Y(I)
      DO 330 KK=1,NTOT
      J=KK+NNOD
      XJ=X(J)
      YJ=Y(J)
      IF(I.EQ.NNOD+1)GO TO 339
      IF(I.EQ.NNOD+2)GO TO 331
      A1=XJ-XX
      B1=YJ-YY
      RJ=DSQRT(A1*A1+B1*B1)
      D=DLOG(RJ)
      CALL CAUCH5(A1,B1,ANG)
      TH=ANG-ANGLE(I)*PI/180.
      IF(TH.LT.0.)TH=TH+2.*PI
      ALPHA=A1*D-TH*B1
      BETA=B1*D+TH*A1
      GO TO 337
331   ALPHA=XJ
      BETA=YJ
      GO TO 337
339   ALPHA=1.
      BETA=0.
337   GO TO(335,345,335)KTYPE(J)
335   G(IIM1,KK)=ALPHA
      G(II,KK)=-1.*BETA
      GO TO 330
345   G(IIM1,KK)=BETA
      G(II,KK)=ALPHA
330   CONTINUE
      IF(KODE.NE.1)GO TO 320
      WRITE(NWT,23)I
      WRITE(NWT,22)(G(IIM1,K),K=1,NTOT)
      WRITE(NWT,22)(G(II,K),K=1,NTOT)
320   CONTINUE
C
C  USE THE GRAM-SCHMIDT ORTHONORANLIZATION PROCESS
C  TO DETERMINE SERIES OF ORTHOGONAL VECTORS
C
      DO 20 I=2,NTOTL
      DO 20 KK=2,I
      SUM1=0.
      SUM2=0.
      DO 30 J=1,NTOT
      SUM1=SUM1+G(I,J)*G(KK-1,J)
      SUM2=SUM2+G(KK-1,J)*G(KK-1,J)
30    CONTINUE
      XXK=-1.*SUM1/SUM2
      XK(I,KK-1)=XXK
      DO 40 J=1,NTOT
      G(I,J)=G(I,J)+XXK*G(KK-1,J)
40    CONTINUE
20    CONTINUE
      IF(KODE.NE.1)GO TO 55
      WRITE(NWT,25)
      DO 47 I=1,NTOTL
47    WRITE(NWT,22)(G(I,J),J=1,NTOT)
```

```
C..CHECK ORTHOGONALITY OF VECTORS G(I)
      WRITE(NWT,26)
55    DO 50 I=1,NTOTL
      IP1=I+1
      IF(I.EQ.NTOTL)GO TO 80
      DO 70 K=IP1,NTOTL
      SUM=0.
      DO 60 J=1,NTOT
60    SUM=SUM+G(I,J)*G(K,J)
      IF(KODE.EQ.1)WRITE(NWT,3)I,K,SUM
      IF(KODE.NE.1 .AND. ABS(SUM).GT..00001)WRITE(NWT,3)I,K,SUM
70    CONTINUE
50    CONTINUE
80    CONTINUE
C..COMPUTE THE COEFFICIENTS OF B(I)=(W,G(I))/(G(I),G(I))
      IF(KODE.EQ.1)WRITE(NWT,28)
      SUM=0.
      DO 120 I=1,NTOTL
      BK1=0.
      BK2=0.
      DO 130 KK=1,NTOT
      J=KK+NNOD
      IF(KTYPE(J).EQ.1 .OR. KTYPE(J).EQ.3)BK1=BK1+VALUE(J,1)*G(I,KK)
      IF(KTYPE(J).EQ.2)BK1=BK1+VALUE(J,2)*G(I,KK)
      BK2=BK2+G(I,KK)*G(I,KK)
130   CONTINUE
C..COMPUTE THE NORM OF THE GENERALIZED FOURIER COEFFICIENTS
      B(I)=BK1/BK2
      SUM=SUM+BK1*BK1/BK2
120   CONTINUE
      IF(KODE.EQ.1)WRITE(NWT,22)(B(I),I=1,NTOTL)
      IF(KADD.NE.1)GO TO 660
      IJ=NODE+1
      WRITE(NTMW,34)X(IJ),Y(IJ),SUM
      WRITE(NWT,34)X(IJ),Y(IJ),SUM
      WRITE(NTMW,35)
      READ FREE(NTMR)KOK
      IF(KOK.NE.1)KOK=0
      IF(KOK.NE.1)GO TO 680
      WRITE(NWT,99)
      WRITE(NWT,34)X(IJ),Y(IJ),SUM
      GO TO 615
C..COMPUTE THE COEFFICIENTS OF THE APPROXIMATE FUNCTIONS
660   DO 200 I=NTOTL,1,-1
      IF(I.EQ.NTOTL)XK(NTOTL,I)=B(NTOTL)
      IF(I.NE.NTOTL)XK(NTOTL,I)=XK(NTOTL,I)*B(NTOTL)+B(I)
200   CONTINUE
      NTOT1=NTOTL-1
      DO 210 I=NTOT1,1,-1
      DO 210 J=I,1,-1
      IF(I.EQ.J)GO TO 210
      IF(I.NE.J)XK(NTOTL,J)=XK(NTOTL,I)*XK(I,J)+XK(NTOTL,J)
210   CONTINUE
      IF(KODE.EQ.1)WRITE(NWT,29)
      IF(KODE.EQ.1)WRITE(NWT,22)(XK(NTOTL,I),I=1,NTOTL)
      DO 110 I=1,NTOT
      B(I)=0.
      WB(I)=0.
110   CONTINUE
C..APPROXIMATE THE EVALUATION POINT -- FUNCTION FORM
      WRITE(NWT,99)
      WRITE(NWT,45)
      DO 480 I=1,NTOT
      II=I+NNOD
      XX=X(II)
      YY=Y(II)
      XRE=0.
      XIM=0.
```

```
      DO 470 J=1,NNODP
      JJ=J*2
      JJM1=JJ-1
      IF(J.EQ.NNOD+1)GO TO 471
      IF(J.EQ.NNOD+2)GO TO 473
      XJ=X(J)
      YJ=Y(J)
      A1=XX-XJ
      B1=YY-YJ
      RJ=DSQRT(A1*A1+B1*B1)
      D=DLOG(RJ)
      CALL CAUCH5(A1,B1,ANG)
      TH=ANG-ANGLE(J)*PI/180.
      IF(TH.LT.0.)TH=2.*PI+TH
      ALPHA=A1*D-TH*B1
      BETA=B1*D+TH*A1
      GO TO 475
471   ALPHA=1.
      BETA=0.
      GO TO 475
473   ALPHA=XX
      BETA=YY
475   XRE=XRE+XK(NTOTL,JJM1)*ALPHA-XK(NTOTL,JJ)*BETA
      XIM=XIM+XK(NTOTL,JJM1)*BETA+XK(NTOTL,JJ)*ALPHA
470   CONTINUE
      XD=XRE-VALUE(II,1)
      IF(KTYPE(II).EQ.2)XD=XIM-VALUE(II,2)
C..MODIFY THE EFFECT BY THE WELL(S)
      IF(NWELL.NE.0) CALL MBC(NWELL,XRE,XIM,XX,YY,2)
      WRITE(NWT,5)I,XRE,XIM,XD
480   CONTINUE
C..APPROXIMATE THE NODAL VALUES -- FUNCTION FORM
290   WRITE(NWT,99)
      WRITE(NWT,4)
      DO 280 I=1,NNOD
      XX=X(I)
      YY=Y(I)
      XRE=0.
      XIM=0.
      DC 270 J=1,NNODP
      IF(I.EQ.J)GO TO 270
      JJ=J*2
      JJM1=JJ-1
      IF(J.EQ.NNOD+1)GO TO 271
      IF(J.EQ.NNOD+2)GO TO 273
      XJ=X(J)
      YJ=Y(J)
      A1=XX-XJ
      B1=YY-YJ
      RJ=DSQRT(A1*A1+B1*B1)
      D=DLOG(RJ)
      CALL CAUCH5(A1,B1,ANG)
      TH=ANG-ANGLE(J)*PI/180.
      IF(TH.LT.0.)TH=2.*PI+TH
      ALPHA=A1*D-TH*B1
      BETA=B1*D+TH*A1
      GO TO 275
271   ALPHA=1.
      BETA=0.
      GO TO 275
273   ALPHA=XX
      BETA=YY
275   XRE=XRE+XK(NTOTL,JJM1)*ALPHA-XK(NTOTL,JJ)*BETA
      XIM=XIM+XK(NTOTL,JJM1)*BETA+XK(NTOTL,JJ)*ALPHA
270   CONTINUE
```

```
C..APPROXIMATE THE NODAL VALUES -- FUNCTION FORM
      IF(NWELL.NE.0) CALL MBC(NWELL,XRE,XIM,XX,YY,2)
      XD=XRE-VALUE(I,1)
      IF(KTYPE(I).EQ.2)XD=XIM-VALUE(I,2)
      WRITE(NWT,5)I,XRE,XIM,XD
280   CONTINUE
      WRITE(NWT,99)
C..BESSEL'S INEQUALITY
      DIFF=SAREA-SUM
      IF(ABS(DIFF).LT.0.00001)DIFF=0.
      WRITE(NWT,36)SAREA,SUM,DIFF
      ERROR=DIFF/SAREA
      WRITE(NTMW,31)
      READ FREE(NTMR)KADD
      IF(KADD.NE.1)KADD=0
      IF(KADD.NE.1)GO TO 260
      IF(KOK.EQ.1)GO TO 690
680   CLOSE NRD
      GO TO 600
690   CLOSE NRD
      OPEN NRD,"LTWO.DAT"
      WRITE FREE(NRD) NNOD,NTEST,KODE,KLIN
      DO 605 I=1,NNOD
      IF(KTYPE(I).EQ.1)VALUE(I,2)=0.
      IF(KTYPE(I).EQ.2)VALUE(I,1)=0.
      WRITE FREE(NRD)X(I),Y(I),KTYPE(I)
      WRITE FREE(NRD)VALUE(I,1),VALUE(I,2),ANGLE(I)
605   CONTINUE
      WRITE FREE(NRD)NWELL
      IF(NWELL.EQ.0) GO TO 607
607   DO 609 J=1,NWELL
609   WRITE FREE(NRD)QX(J),QY(J),Q(J)
      CLOSE NRD
      KOK=0
      GO TO 600
260   WRITE(NTMW,71)
      READ FREE(NTMR) KEVA
      IF(KEVA.NE.1) GO TO 700
      OPEN NWD,"LTWO1.DAT"
      WRITE FREE(NWD) NNOD,NNODP,NTOTL
      DO 710 I=1,NNOD
710   WRITE FREE(NWD) X(I),Y(I),ANGLE(I)
      WRITE FREE(NWD) (XK(NTOTL,I),I=1,NTOTL)
      WRITE FREE(NWD) NWELL
      IF(NWELL.EQ.0) GO TO 720
      DO 730 I=1,NWELL
730   WRITE FREE(NWD) QX(I),QY(I),Q(I)
720   CLOSE NWD
      WRITE(NTMW,72)
700   CLOSE NRD
      CLOSE NWT
      STOP
      END
```

```
      SUBROUTINE MBC(NWELL,XRE,XIM,X,Y,KODE)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/BLK 7/ QX(10),QY(10),Q(10)
C
C  THIS SUBROUTINE MODIFES THE BOUNDARY CONDITION AND THE
C  APPROXIMATE SOLUTION ACCORDING TO THE WELL(S) EFFECT.
C
      TPI=6.283185307
      SGN=1.
      IF(KODE.NE.1)SGN=-1.
      DO 10 J=1,NWELL
      XX=X-QX(J)
      YY=Y-QY(J)
      RJ=DSQRT(XX*XX+YY*YY)
      D=DLOG(RJ)
      XRE=XRE-SGN*(Q(J)*D)/TPI
      CALL CAUCH5(XX,YY,B)
      XIM=XIM-SGN*(Q(J)*B)/TPI
10    CONTINUE
      RETURN
      END
C
```

```
C-----------------------------------------------------------------------
C     SUBROUTINE CAUCH5
C-----------------------------------------------------------------------
      SUBROUTINE CAUCH5(X,Y,ANGLE)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
C
C  THIS SUBROUTINE DETERMINES THE POSITIVE ANGLE
C  OF COMPLEX POINT Z WITH RESPECT TO THE ORIGIN
C
      PI=3.141592653
      IF(X.EQ.0. .AND. Y.GT.0.)ANGLE=.5*PI
      IF(X.EQ.0. .AND. Y.LT.0.)ANGLE=1.5*PI
      IF(X.GT.0. .AND. Y.GE.0.)ANGLE=DATAN(Y/X)
      IF(X.LT.0. .AND. Y.GE.0.)ANGLE=PI-DATAN(-Y/X)
      IF(X.LT.0. .AND. Y.LT.0.)ANGLE=PI+DATAN(Y/X)
      IF(X.GT.0. .AND. Y.LT.0.)ANGLE=2.*PI-DATAN(-Y/X)
      IF(X.EQ.0. .AND. Y.EQ.0.)ANGLE=0.
      RETURN
      END
C
```

7.6.3. Input Variable Description for CVBEM1

The input file has the following form:

Line	Variables	
1	NNOD,NTEST,KODE,KLIN	
2	X(I),Y(I),KTYPE(I),VALUE(I,1),VALUE(I,2), ANGLE(I)	NNOD
⋮		
NNOD+1		
NNOD+2	NWELL	
NNOD+3	QX(I),QY(I),Q(I)	NWELL
NNOD+2+NWELL		

where

NNOD is the total node number.

NTEST is the number of evaluation points per element.

KODE = { 0, Summary output; 1, Detail output }

KLIN = { 0, excludes the linear part of the approximation function; 1, includes the linear part of the approximation function }

X(I) is the x-coordinate

Y(I) is the y-coordinate

KTYPE(I) =
- 1 , state variable specified for Node I
- 2 , stream function specified for Node I
- 3 , state variable and stream function specified for Node I
- 4 , efflux boundary condition

VALUE(I,1) is the value of state variable for Node I

VALUE(I,2) is the value of stream function for Node I

ANGLE(I) is the outer normal for Node I

NWELL is the number of sink or source terms

QX(I) is the x-coordinate for the I^{th} sink or source term

QY(I) is the y-coordinate for the I^{th} sink or source term

Q(I) is the strength of the sink (+) or source (-) term

Note that all the sources and sinks have a branch cut parallel to the real x-axis.

7.6.4. CVBEM2 Program Listing

Figure 7.8 depicts the simple flow chart of the CVBEM2 program and the listing of the program is included in this section.

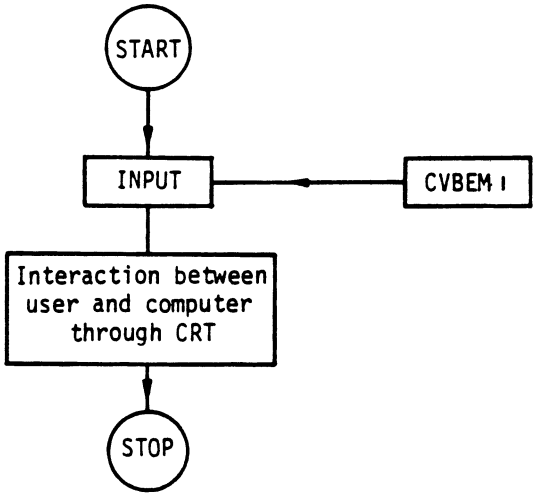


Fig. 7.8. Simple Flow Chart for Program CVBEM2

```
C
C  MAIN PROGRAM
C
C  THIS PROGRAM EVALUATES THE STATE VARIABLE AND STREAM FUNCTION
C  FOR A GIVEN NODAL POINT
C
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      COMMON/BLK 1/ X(50),Y(50)
      COMMON/BLK 2/ ANGLE(50),XK(50)
      COMMON/BLK 7/ QX(10),QY(10),Q(10)
C
C  OPEN DATA FILES
C
      NRD=1
      NWD=2
      NTR=11
      NTW=10
      OPEN NRD,"LTWO1.DAT"
C
C READ INPUT DATA
C
      PI=3.141592653
      READ FREE(NRD) NNOD,NNODP,NTOTL
      DO 10 I=1,NNOD
      READ FREE(NRD) X(I),Y(I)
10    CONTINUE
      READ FREE(NRD) (XK(I),I=1,NTOTL)
      READ FREE(NRD) NWELL
      IF(NWELL.EQ.0) GO TO 15
      DO 17 I=1,NWELL
17    READ FREE(NRD) QX(I),QY(I),Q(I)
15    CLOSE (NRD)
      DO 18 I=1,NNOD
      WRITE(10,60)I
60    FORMAT(/,2X,'ENTER THE BRANCH CUT FOR NODE    ,I3)
      READ FREE(11)ANGLE(I)
      ANGLE(I)=ANGLE(I)*PI/180.
18    CONTINUE
C
C  OUTPUT FORMATS
C
4     FORMAT(/,2X,'*** INVALID DATA ENTRY...TRY AGAIN ***',/)
5     FORMAT(3X,F10.4,2X,F10.4,2X,2(2X,F10.3))
6     FORMAT(10X,'*** APPROXIMATE SOLUTION ***',/)
7     FORMAT(2X,'EVALUATE THE STATE VARIABLE AND STREAM FUNCTION',
     1' FOR A GIVEN POINT',/,2X,'ENTER THE X-COORDINATE')
8     FORMAT(2X,'ENTER THE Y-COORDINATE')
9     FORMAT(/,7X,'X-',10X,'Y-',10X,'STATE',6X,'STREAM',
     1/,3X,'COORDINATE',2X,'COORDINATE',4X,'VARIABLE',4X,
     2'FUNCTION',/)
11    FORMAT(/,2X,'ENTER THE OPTION :',/,3X,
     1'[1] FOR ANOTHER POINT',/,3X,'[2] FOR ACCEPTING THE'
     2,' CURRENT POINT AND STARTING A NEW POINT',/,3X,
     3'[3] FOR ACCEPTING THE CURRENT POINT AND TERMINATING'
     4,' THE PROCESS')
12    FORMAT(/,2X,'*** THE NODAL POINT HAS BEEN USED...TRY AGAIN'
     1,' ***',/)
```

```
C
C  APPROXIMATE THE STATE VARIABLE AND STREAM FUNCTION
C
      OPEN NWD,"LTWO1.ANS"
      WRITE(NWD,6)
      WRITE(NWD,9)
300   WRITE(NTW,7)
      READ FREE(NTR) XX
      WRITE(NTW,8)
      READ FREE(NTR) YY
      DO 100 I=1,NNOD
      IF(XX.EQ.X(I) .AND. YY.EQ.Y(I))GO TO 110
100   CONTINUE
      GO TO 120
110   WRITE(NTW,12)
      GO TO 300
120   XRE=0.
      XIM=0.
      DO 270 J=1,NNODP
      JJ=J*2
      JJM1=JJ-1
      IF(J.EQ.NNOD+1)GO TO 271
      IF(J.EQ.NNOD+2)GO TO 273
      A1=XX-X(J)
      B1=YY-Y(J)
      RJ=DSQRT(A1*A1+B1*B1)
      D=DLOG(RJ)
      CALL CAUCH5(A1,B1,ANG)
      TH=ANG-ANGLE(J)*PI/180.
      IF(TH.LT.0.)TH=2.*PI+TH
      ALPHA=A1*D-TH*B1
      BETA=B1*D+TH*A1
      GO TO 275
271   ALPHA=1.
      BETA=0.
      GO TO 275
273   ALPHA=XX
      BETA=YY
275   XRE=XRE+XK(JJM1)*ALPHA-XK(JJ)*BETA
      XIM=XIM+XK(JJM1)*BETA+XK(JJ)*ALPHA
270   CONTINUE
C..MODIFY THE EFFECT BY THE WELL(S)
      IF(NWELL.NE.0) CALL MBC(NWELL,XRE,XIM,XX,YY,2)
      WRITE(NTW,9)
      WRITE(NTW,5)XX,YY,XRE,XIM
330   WRITE(NTW,11)
      READ FREE(NTR) KODE
      IF(KODE.EQ.1) GO TO 300
      IF(KODE.EQ.2) GO TO 310
      IF(KODE.EQ.3) GO TO 320
      WRITE(NTW,4)
      GO TO 330
310   WRITE(NWD,5)XX,YY,XRE,XIM
      GO TO 300
320   WRITE(NWD,5)XX,YY,XRE,XIM
      CLOSE NWD
      STOP
      END
```

```
      SUBROUTINE MBC(NWELL,XRE,XIM,X,Y,KODE)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/BLK 7/ QX(10),QY(10),Q(10)
C
C  THIS SUBROUTINE MODIFES THE BOUNDARY CONDITION AND THE
C  APPROXIMATE SOLUTION ACCORDING TO THE WELL(S) EFFECT.
C
      TPI=6.283185307
      SGN=1.
      IF(KODE.NE.1)SGN=-1.
      DO 10 J=1,NWELL
      XX=X-QX(J)
      YY=Y-QY(J)
      RJ=DSQRT(XX*XX+YY*YY)
      D=DLOG(RJ)
      XRE=XRE-SGN*(Q(J)*D)/TPI
      CALL CAUCH5(XX,YY,B)
      XIM=XIM-SGN*(Q(J)*B)/TPI
10    CONTINUE
      RETURN
      END
C
```

```
C-----------------------------------------------------------------------
C     SUBROUTINE CAUCH5
C-----------------------------------------------------------------------
      SUBROUTINE CAUCH5(X,Y,ANGLE)
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
C
C  THIS SUBROUTINE DETERMINES THE POSITIVE ANGLE
C  OF COMPLEX POINT Z WITH RESPECT TO THE ORIGIN
C
      PI=3.141592653
      IF(X.EQ.0. .AND. Y.GT.0.)ANGLE=.5*PI
      IF(X.EQ.0. .AND. Y.LT.0.)ANGLE=1.5*PI
      IF(X.GT.0. .AND. Y.GE.0.)ANGLE=DATAN(Y/X)
      IF(X.LT.0. .AND. Y.GE.0.)ANGLE=PI-DATAN(-Y/X)
      IF(X.LT.0. .AND. Y.LT.0.)ANGLE=PI+DATAN(Y/X)
      IF(X.GT.0. .AND. Y.LT.0.)ANGLE=2.*PI-DATAN(-Y/X)
      IF(X.EQ.0. .AND. Y.EQ.0.)ANGLE=0.
      RETURN
      END
C
```

EXAMPLE 7.4: (Example Input-Output Data)

Approximate the analytical function $\omega = e^z$ on a π-square region. Figure 7.9 shows the nodal points placement and depicts the boundary conditions.

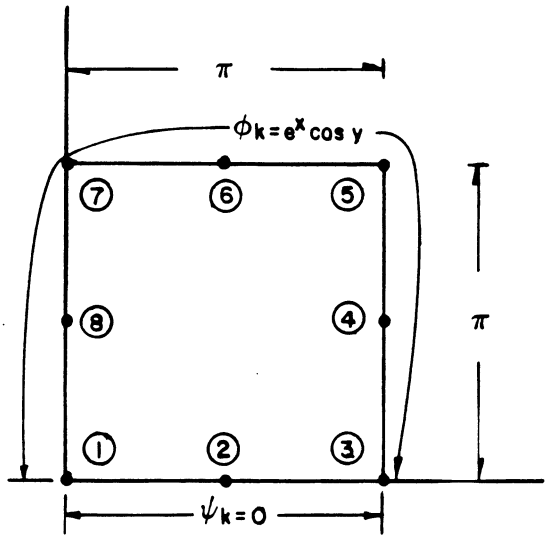


Fig. 7.9. Example Input-output Problem ($\omega = e^z$) for Program CVBEM1

The input data file is stored in "LTWO.DAT" which consists of:

```
8   5   1   1
0   0   3   1   0   270
1.570796 0  2  0  0  270
3.141592 0  3  23.14069  0 0
3.141592  1.570796  1  0 0 0
3.141592  3.141592  1  -23.14069  0  90
```

```
1.570796    3.141592    1    -4.81048    0    90
   0        3.141592    1       -1       0   180
   0        1.570796    1        0       0   180
   0
END OF FILE
```

Output File Description

The output file is stored in "LTWO.ANS" which has the following form:

(a) Boundary Node Data

(b) Evaluation Point Data

(c) Nodal Point Vector Expansion (optional)

The vector is filled by the real or imaginary part of the approximate function ($C_{m+1} + C_{m+2}\, z + \sum_{j=1}^{m} (z - z_j)\mathrm{Ln}(z - z_j)$ where m is the total node number) according to the boundary conditions

(d) Orthogonal Vector Expansion (Optional)

Using the Gram-Schmidt orthonormalization process to determine series of orthogonal vectors of part (c)

(e) Orthogonal Test (Optional)

(f) Evaluation Coefficients (Optional)

Coefficients for orthogonal vector of part (d)

(g) Back Substitution Coefficients (Optional)

Coefficients for original vector of part (c)

(h) Approximation Point Values and Errors

The error is defined as the difference between the the approximation value and the specified boundary condition

(i) Approximation Nodal Values and Errors

(j) Bessel's Inequality

The left hand side of the inequality sign is the dot product of the boundary condition vector and the right hand side is the dot product of the normalized Fourier coefficients

The output file is included hereafter.

*** BOUNDARY NODE DATA ***

NODE NO.	X(I)	Y(I)	KTYPE(I) 1=SV;2=SF 3=SV&SF	VAUE(I) SV	SF	OUTER NORMAL ANGLE(I)
1	.000	.000	3	1.00	.00	270.00
2	1.571	.000	2	.00	.00	270.00
3	3.142	.000	3	23.14	.00	.00
4	3.142	1.571	1	.00	.00	.00
5	3.142	3.142	1	-23.14	.00	90.00
6	1.571	3.142	1	-4.81	.00	90.00
7	.000	3.142	1	-1.00	.00	180.00
8	.000	1.571	1	.00	.00	180.00

*** EVALUATION POINT DATA ***

POINT NO.	X(I)	Y(I)	KTYPE(I) 1=SV;2=SF 3=SV&SF	VALUE(I) SV	SF
1	.131	.000	2	.00	.00
2	.458	.000	2	.00	.00
3	.785	.000	2	.00	.00
4	1.113	.000	2	.00	.00
5	1.440	.000	2	.00	.00
6	1.702	.000	2	.00	.00
7	2.029	.000	2	.00	.00
8	2.356	.000	2	.00	.00
9	2.683	.000	2	.00	.00
10	3.011	.000	2	.00	.00
11	3.142	.131	1	21.21	.00
12	3.142	.458	1	16.39	.00
13	3.142	.785	1	11.57	.00
14	3.142	1.113	1	6.75	.00
15	3.142	1.440	1	1.93	.00
16	3.142	1.702	1	-1.93	.00
17	3.142	2.029	1	-6.75	.00
18	3.142	2.356	1	-11.57	.00
19	3.142	2.683	1	-16.39	.00
20	3.142	3.011	1	-21.21	.00
21	3.011	3.142	1	-21.61	.00
22	2.683	3.142	1	-17.79	.00
23	2.356	3.142	1	-13.98	.00
24	2.029	3.142	1	-10.16	.00
25	1.702	3.142	1	-6.34	.00
26	1.440	3.142	1	-4.49	.00
27	1.113	3.142	1	-3.70	.00
28	.785	3.142	1	-2.91	.00
29	.458	3.142	1	-2.11	.00
30	.131	3.142	1	-1.32	.00
31	.000	3.011	1	-.92	.00
32	.000	2.683	1	-.71	.00
33	.000	2.356	1	-.50	.00
34	.000	2.029	1	-.29	.00
35	.000	1.702	1	-.08	.00
36	.000	1.440	1	.08	.00
37	.000	1.113	1	.29	.00
38	.000	.785	1	.50	.00
39	.000	.458	1	.71	.00
40	.000	.131	1	.92	.00

*** EVALUATION COEFFICIENTS. (G(I),B(I))/(G(I),G(I)) ***

73088 00	35110 00	- 50280 00	14100 01	38110 01	31420 00	66780 01	- 38080 01	47120 01	54610 01
46440 00	15100 01	24770 00	12040 01	15780 00	63380 00	- 49810 02	27680 02	15840 02	- 15220 02

*** BACKSUBSTITUTION COEFFICIENTS ***

74280 01	12690 01	51750-02	91430 00	- 60690 01	47600 01	13000 00	- 41750 01	67630 01	44860 01
12890 00	75110 00	43010 02	91360 00	- 34440-01	39810 00	- 10270 03	27470 02	18570 02	- 15220 02

*** APPROXIMATE POINT VALUES AND ERRORS -- FUNCTION FORM ***

POINT NUMBER	STATE VARIABLE	STREAM FUNCTION	ERROR
1	.1049D+01	-.4281D-01	-.4281D-01
2	.1221D+01	.4437D-01	.4437D-01
3	.1583D+01	.2733D-01	.2733D-01
4	.2213D+01	-.1657D-01	-.1657D-01
5	.3192D+01	-.3405D-01	-.3405D-01
6	.4227D+01	.2142D-01	.2142D-01
7	.5986D+01	.1089D-01	.1089D-01
8	.8560D+01	-.6202D-02	-.6202D-02
9	.1237D+02	-.7143D-02	-.7143D-02
10	.1854D+02	.2761D-02	.2761D-02
11	.2123D+02	.4625D+01	.1876D-01
12	.1638D+02	.1076D+02	-.1425D-01
13	.1155D+02	.1436D+02	-.2333D-01
14	.6753D+01	.1655D+02	.3151D-02
15	.1975D+01	.1772D+02	.4638D-01
16	-.1955D+01	.1801D+02	-.2686D-01
17	-.6792D+01	.1746D+02	-.4276D-01
18	-.1156D+02	.1612D+02	.1343D-01
19	-.1633D+02	.1401D+02	.6167D-01
20	-.2125D+02	.1107D+02	-.4091D-01
21	-.2165D+02	.7595D+01	-.3852D-01
22	-.1772D+02	.3666D+01	.7063D-01
23	-.1396D+02	.6227D+00	.1725D-01
24	-.1021D+02	-.1509D+01	-.5343D-01
25	-.6378D+01	-.2406D+01	-.3996D-01
26	-.4418D+01	-.1276D+01	.7529D-01
27	-.3689D+01	-.1148D+01	.1050D-01
28	-.2950D+01	-.1462D+01	-.4524D-01
29	-.2147D+01	-.1844D+01	-.3587D-01
30	-.1250D+01	-.2101D+01	.6790D-01
31	-.9230D+00	-.1791D+01	-.6312D-02
32	-.7570D+00	-.1168D+01	-.4862D-01
33	-.5017D+00	-.5855D+00	-.1747D-02
34	-.2471D+00	-.7380D-01	.4455D-01
35	-.6163D-01	.3229D+00	.2171D-01
36	.1849D-01	.3965D+00	-.6485D-01
37	.2927D+00	.3766D+00	.1055D-02
38	.5445D+00	.3071D+00	.4449D-01
39	.7331D+00	.1706D+00	.2477D-01
40	.8778D+00	-.4606D-01	-.3888D-01

*** APPROXIMATE NODAL VALUES AND ERRORS ***

NODE NUMBER	STATE VARIABLE	STREAM FUNCTION	ERROR
1	.9910D+00	-.1845D+00	-.8966D-02
2	.3701D+01	-.3379D-02	-.3379D-02
3	.2316D+02	-.1303D-01	.1451D-01
4	.3716D-01	.1797D+02	.3716D-01
5	-.2341D+02	.9531D+01	-.2689D+00
6	-.4762D+01	-.1937D+01	.4874D-01
7	-.8569D+00	-.2048D+01	.1431D+00
8	-.5330D-01	.4180D+00	-.5330D-01

*** BESSEL INEQUALITY ***
.2978D+04 >= .2978D+04 AND THE DIFFERENCE IS .5624D-01

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APPENDIX A: DERIVATION OF CVBEM APPROXIMATION FUNCTION

Let $\omega \in W_{\Omega}^{A}$, and P_n be a nodal partition of Γ. Define a global trial function $G_m(\zeta)$ on Γ by

$$G_m(\zeta) = \sum_{j=1}^{m} N_j(\zeta)\omega_j$$

where $\omega_j = \omega(z_j)$ and $\zeta \in \Gamma$. Develop the integral function $A(z)$ defined by

$$A(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G_m(\zeta)d\zeta}{\zeta - z}, \quad z \in \Omega$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{G_m(\zeta)d\zeta}{\zeta - z}$$

On Γ_j, $G_m(\zeta) = \omega_j \left[(z_{j+1} - \zeta)/(z_{j+1} - z_j)\right]$

$$+ \omega_{j+1} \left[(\zeta - z_j)/(z_{j+1} - z_j)\right]$$

and

$$\int_{\Gamma_j} \frac{G_m(\zeta)d\zeta}{\zeta - z} = \omega_{j+1} \left[1 + \left(\frac{z - z_j}{z_{j+1} - z_j}\right)\left(\text{Ln}\,(z_{j+1} - z) - \text{Ln}\,(z_j - z)\right)\right]$$

$$- \omega_j \left[1 + \left(\frac{z - z_{j+1}}{z_{j+1} - z_j}\right)\left(\text{Ln}\,(z_{j+1} - z) - \text{Ln}\,(z_j - z)\right)\right]$$

Summing from j = 1 to m, (and noting $\omega_{m+1} = \omega_1$ and $z_{m+1} = z_1$)

$$\mathcal{X} = \sum_{j=1}^{m} \int_{\Gamma_j} \frac{G_m(\zeta)d\zeta}{\zeta - z} = \sum_{j=1}^{m} (\omega_{j+1} - \omega_j)$$

$$+ \sum_{j=1}^{m} \frac{\left[(\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1}))\,(\mathrm{Ln}\,(z_{j+1} - z) - \mathrm{Ln}(z_j - z))\right]}{(z_{j+1} - z_j)}$$

where $\mathrm{Ln}(z_{m+1} - z) = \mathrm{Ln}(z_1 - z) + 2\pi i$.

Thus

$$\mathcal{X} = \sum_{j=1}^{m-1} \left[\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1})\right]\left[\mathrm{Ln}(z_{j+1} - z) - \mathrm{Ln}(z_j - z)\right]/(z_{j+1} - z_j)$$

$$+ \left[\omega_1(z - z_m) - \omega_m(z - z_1)\right]\left[\mathrm{Ln}(z_1 - z) + 2\pi i - \mathrm{Ln}(z_m - z)\right]/(z_1 - z_m)$$

Combining terms with respect to the $\mathrm{Ln}(z_j - z)$ functions gives

$$\mathcal{X} = \sum_{j=1}^{m} \left\{ \frac{[\omega_j(z - z_{j-1}) - \omega_{j-1}(z - z_j)]}{(z_j - z_{j-1})} - \frac{[\omega_{j+1}(z - z_j) - \omega_j(z - z_{j+1})]}{(z_{j+1} - z_j)} \right\} \mathrm{Ln}(z_j - z)$$

$$+ 2\pi i \frac{[\omega_1(z - z_m) - \omega_m(z - z_1)]}{(z_1 - z_m)}$$

Thus if $P_j(\zeta)$ is the interpolation function on Γ_j given by

$$P_j(\zeta) = \begin{cases} N_j(\zeta)\omega_j + N_{j+1}(\zeta)\omega_{j+1}, & \zeta \in \Gamma_j \\ 0, & \text{otherwise} \end{cases}$$

then by substituting z into ζ of $P_j(\zeta)$

$$\chi = \sum_{j=1}^{m} (P_{j-1}(z) - P_j(z)) \mathrm{Ln}(z_j - z) + 2\pi i \, P_m(z)$$

where now $P_o(z) \equiv P_m(z)$. Finally, $P_j(z_j) = P_{j-1}(z_j)$ implies that

$$A(z) = P_m(z) + \frac{1}{2\pi i} \sum_{j=1}^{m} \left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})} \right] (z_j - z) \, \mathrm{Ln} \, (z_j - z)$$

Complex constants k_j can be used to simplify the writing of A(z) by

$$A(z) = P_m(z) + \sum_{j=1}^{m} k_j \, (z_j - z) \, \mathrm{Ln} \, (z_j - z)$$

where

$$k_j = \frac{1}{2\pi i} \left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})} \right]$$

Noting $\mathrm{Ln} \, (z_j - z) = \mathrm{Ln} \, (z - z_j) + \mathrm{Ln} \, (-1) = \mathrm{Ln} \, (z - z_j) + i\pi$, A(z) can be rewritten as

$$A(z) = P_m(z) - \sum_{j=1}^{m} k_j (z - z_j) \, \mathrm{Ln} \, (z - z_j) + i\pi \sum_{j=1}^{m} k_j (z_j - z)$$

In the above, $\mathrm{Ln} \, (z - z_j)$ is measured with respect to point $z \in \Omega$ as the branch point. It is desirable to define branch points at each z_j with branch cuts lying exterior of Ω. This process introduces an additional angle term θ^j of the branch cut for each node such that

$$\mathrm{Ln} \, (z - z_j) = \mathrm{Ln}_j \, (z - z_j) + i\theta^j$$

where Ln_j is notation of individual logarithm functions.
Thus A(z) is of the form

$$A(z) = R_1(z) + \sum_{j=1}^{m} c_j (z - z_j) \, \mathrm{Ln}_j (z - z_j)$$

where $R_1(z)$ is a first degree complex polynomial

$$R_1(z) = P_m(z) + \sum_{j=1}^{m} i(\pi + \theta^j)\, k_j\, (z_j - z)$$

and each $c_j = - k_j$.

APPENDIX B: CONVERGENCE OF CVBEM APPROXIMATOR

Let $\omega \in W_{\Omega}^{A}$ and $z \in \Omega$.

Let $A(z) = P_m(z) + \sum_{j=1}^{m} k_j (z_j - z) \operatorname{Ln} (z_j - z)$,

where $k_j = \frac{1}{2\pi i} \left[\frac{(\omega_{j+1} - \omega_j)}{(z_{j+1} - z_j)} - \frac{(\omega_j - \omega_{j-1})}{(z_j - z_{j-1})} \right]$

and $P_m(z) = \omega_1 + \left[\frac{\omega_1 - \omega_m}{z_1 - z_m} \right] (z - z_1)$.

Because $\omega(z)$ is analytic on $\bar{\Omega}$, ℓ can be chosen small enough such that any three neighboring (in sequence) nodes z_{j-1}, z_j, z_{j+1} lie within the radius of convergence of the Taylor series about z_j. That is,

$$\omega_{j+1} = \omega_j + \omega_j'(z_{j+1} - z_j) + \omega_j''(z_{j+1} - z_j)^2/2! + \cdots$$

$$\omega_{j-1} = \omega_j - \omega_j'(z_j - z_{j-1}) + \omega_j''(z_j - z_{j-1})^2/2! + \cdots$$

Thus

$$(\omega_{j+1} - \omega_j)/(z_{j+1} - z_j) = \omega_j' + \omega_j''(z_{j+1} - z_j)/2! + \cdots$$

$$(\omega_{j-1} - \omega_j)/(z_j - z_{j-1}) = -\omega_j' + \omega_j''(z_j - z_{j-1})/2! + \cdots$$

and

$$k_j = \frac{1}{2\pi i} \left[\omega_j'' \frac{(z_{j+1} - z_{j-1})}{2} + r_j \right]$$

where r_j is the residual terms of the Taylor series such that $r_j \to 0$ in order 2 as $\ell \to 0$. Thus

$$\lim_{\substack{m\to\infty \\ \ell\to 0}} A(z) = \lim_{\substack{m\to\infty \\ \ell\to 0}} \left[P_m(z) + \sum_{j=1}^{m} \frac{1}{2\pi i}\, \omega_j'' \, \frac{(z_{j+1} - z_{j-1})}{2} (z_j - z) \operatorname{Ln} (z_j - z) \right.$$

$$\left. + \sum_{j=1}^{m} \frac{1}{2\pi i}\, r_j (z_j - z) \operatorname{Ln} (z_j - z) \right]$$

Evaluating terms,

$$\lim_{\substack{m\to\infty \\ \ell\to 0}} P_m(z) = \lim_{\ell\to 0} \left[\omega_1 + \left(\frac{\omega_1 - \omega_m}{z_1 - z_m} \right) (z - z_1) \right] = \omega_1 + \left. \frac{d\omega}{d\zeta} \right|_{z_1} (z - z_1)$$

and therefore

$$\lim_{\substack{m\to\infty \\ \ell\to 0}} A(z) = \omega_1 + \left. \frac{d\omega}{d\zeta} \right|_{z_1} (z - z_1) + \frac{1}{2\pi i} \int_{\Gamma} \frac{d^2\omega}{d\zeta^2} (\zeta - z) \operatorname{Ln} (\zeta - z) dz$$

Integrating by parts,

$$\int_{\Gamma} \frac{d^2\omega}{d\zeta^2} (\zeta - z) \operatorname{Ln} (\zeta - z)\, dz = (\zeta - z) \operatorname{Ln} (\zeta - z) \left. \frac{d\omega}{d\zeta} \right|_{\Gamma}$$

$$- \int_{\Gamma} \frac{d\omega}{d\zeta} (1 + \operatorname{Ln} (\zeta - z))\, d\zeta$$

where

$$(\zeta - z) \operatorname{Ln} (\zeta - z) \left. \frac{d\omega}{d\zeta} \right|_{\Gamma} = 2\pi i (z_1 - z) \left. \frac{d\omega}{d\zeta} \right|_{z_1}$$

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